# An Exact Connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs *†

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**Abstract**

The paper proposes a new approach to consistent stochastic utilities, also called forward dynamic utilities, recently introduced by M. Musiela and T. Zariphopoulou [26]. These utilities satisfy a property of consistency with a given incomplete financial market which provides them properties similar to the function values of classical portfolio optimization. The additional assumption of the existence of optimal wealth plays a key role in this paper. Using Itô-Ventzel formula, we derive two forward non linear stochastic PDEs of HJB type satisfied by consistent

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stochastic utilities processes of Itô type and their dual convex conjugates. We characterize the volatility of consistent utilities as an operator of the first and the second order derivatives of the utility in terms of the optimal primal and dual policies. Making the assumption that the SDEs, associated with the optimal policies, are solvable with monotone solutions with respect to their initial conditions, we characterize all consistent utilities for a given increasing optimal wealth process as the primitive of the composition of the dual optimal process and the inverse of the optimal wealth. Also, we establish a connection between the fully nonlinear second order utility SPDE and two stochastic differential equations associated with optimal policies, which allows us to reduce the resolution of the utility HJB-SPDE to the existence of monotone solutions of the SDEs.

# Introduction

Recently, the concept of forward dynamic utilities has been introduced by M. Musiela and T. Zariphopoulou (2004-2007) [26, 29, 27, 30, 32, 34], to model possible changes over the time of individual preferences of an agent. Such a concept has also been studied by F. Berrier, M. Tehranchi and C.Rogers (2009) [10] and G. Zitkovic [46]. Further works related to this problem are T. Choulli, C. Stricker and L. Jia (2007) [3], V. Henderson and D. Hobson (2007) [13].

The economic agent will adjust its preferences based on the information that is revealed over time and represented by a filtration $(\mathcal{F}_t, t \geq 0)$, defined on the probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t, t \geq 0))$. In contrast to the classical literature, there is no pre-specified trading horizon at the end of which the utility datum is assigned. Rather, the agent starts with today's specification of its utility, $U(0, x) = u(x)$, and then builds the process $U(t, x)$ for $t > 0$ in relation to the information flow given by $(\mathcal{F}_t, t \geq 0)$. This, together with the choice of a initial utility, distinguishes the forward dynamic utility from the recursive utility for which the aggregator can be specified exogenously and the value function is recovered backward in time.

Working with positive wealth processes $X^\pi$ in an incomplete market, we define a consistent stochastic utility as a progressive non negative stochastic utility $U(t, x)$, for which $U(t, X_t^\pi)$ is a supermartingale, and a martingale for one optimal wealth. However we restrict our study to forward utilities which are Itô-semimartingales with spatial parameter $x$, whose local charac-

teristics $(\beta, \gamma)$ are such that

$$dU(t,x) = \beta(t,x)dt + \gamma(t,x).dW_t.$$

The key tool is the Itô-Ventzel's formula which we recall at the beginning of Section 2. As in the classical Hamilton-Jacobi-Bellman framework, we proceed by verification to establish the dynamics of consistent utilities. Assuming a sufficient constraint on the drift $\beta$ of HJB type, we get the utility stochastic PDE that we investigate in this paper. In particular, we study the role of the utility risk premium defined by $\eta^U_t(x) = \gamma_x/U_x(t,x)$. Section 3 goes into details on the question of duality and gives a characterization of the non linear SPDE satisfied by the progressive convex conjugate $\tilde{U}$ of $U$. This allows us to obtain a complete interpretation of the volatility $\gamma$. Unlike the backward case, we do not a priori give a positive answer on the question of existence and uniqueness of solutions of the primal and dual Hamilton-Jacobi-Bellman equations established in this work, but show the important role of the volatility $\gamma$ of the stochastic utility $U$ and the strong analogy between the primal and dual problem. The obstacles in the analysis come from the fact that the HJB equations are forward in time and also without maximum principle. Therefore existing results of existence, uniqueness and regularity of weak (viscosity) solutions are not directly applicable. An additional difficulty comes from the fact that the volatility random field may depend on higher order derivatives of $U$, in which case the SPDE can not be turned into a regular PDE with random coefficients, using the method of stochastic characteristics. Moreover, the concavity property can not be derived directly from the dynamics; this still an open question in general, which we try to answer in Section 4. In Section 3, we focus on the convex conjugate function $\tilde{U}(t,y)$ of $U(t,x)$. We show that this conjugate random fields is a solution of a dual Utility-SPDE and is consistent with the familly of state price density processes, in particular, there exists an optimal choice $Y^*$ which plays an important role in this paper. In Section 3.3, we show the stability of the notion of consistent utility by change of numeraire and then, without loss of generality, we can consider the martingale market where the portfolios are simple local martingales and the stochastic PDE's are easier to deal with.

In Section 4, we establish the most original contribution of this paper, that is a new approach to consistent dynamic utilities based on the stochastic flows associated with the optimal wealth and the optimal state price process and their inverses. The idea is very simple and natural: Suppose that the optimal portfolio denoted by $X^*_t(x)$ is strictly increasing with respect to the initial capital, and denote by $(\mathcal{X}(t,x))$ the adapted inverse process, defined by $X^*_t(\mathcal{X}(t,x)) = x$.

Then, using the dual identity $U_x(t, X_t^*(x)) = Y_t^*(u_x(x))$, we can find $U_x(t,x)$ from $U_x(t,x) = \mathcal{Y}(t, \mathcal{X}(t,x))$ where $(Y_t^*(y))$ is the optimal state price density process and $\mathcal{Y}(t,x) := Y_t^*(u_x(x))$. Finally we get $U$ by integration. So, we are able to generate all the consistent utilities with a given optimal portfolio.
The problem of recovering the utility function coherent with a given optimal portfolio is known in the financial literature as the "inverse" Merton problem; it has been considered by many authors in the past in particular by H.He and C.Huang (1992), [12]. In the classical expected optimization problem, there are restrictions to put so that the portfolio is consistent with a deterministic utility criterion at some fixed time horizon. These difficulties disappear when the criterion is a progressive utility as we show in this paper.
The study provides a fine analysis of the utility volatility vector and its derivative in terms of optimal allocation policy and optimal choice of state price density. In fact, given these optimal policies, the volatility vector $\gamma$ is interpreted as an operator $\Upsilon(x, U_x, U_{xx})$ which is linear in $U_{xx}$ and depend on $U_x$ (resp. $x$) as how the volatility of the flow $\mathcal{Y}$ (resp. $X^*$) depends on $\mathcal{Y}$ (resp. $X^*$).
To the best of our knowledge, the fully non linear utility stochastic PDE's established in this paper and satisfied by forward utilities and their dual have not been established in a general way. In [10] and [35] the authors study the case where the volatility vector of the utility is zero. In [32], the authors derive a stochastic PDE and study examples where the volatility of the utility is constant, proportional to $U$ (case of change of probability) and the case where the volatility is proportional to $xU_x$ which corresponds to a change of numeraire.
Furthermore, to our knowledge, there is no general consistent utilities construction proposed in the literature, expect the case of power or exponential type, or decreasing utilities.

Another main contribution of this paper is a connection between two solvable SDEs and the utility SPDEs early established. In particular, given a volatility vector $\gamma$ such that $\gamma_x(t,x) = -xU_{xx}\kappa_t^*(x) + \nu_t^*(U_x(t,x))$, we show the existence and uniqueness of a solution to the fully nonlinear second order SPDE from that of a pair of SDE's. In any case this represents an interesting result in the theory of stochastic partial differential equations.

The paper is organized as follows, we give the definition of consistent dynamic utilities. Then, in order to study the HJB Stochastic PDE, we give more precisions on the market model. In

Section 2, we introduce the useful Itô-Ventzel formula and we provide the dynamics of consistent utilities and a closed form for the optimal policy and we give an example of consistent utility obtained by combining a standard utility function with some positive processes. In Section 3, we study the dual process and establish a duality identity. In Section 3.3, we show the stability of the notion of consistent utility by a change of numeraire. In Section 4, we present our new approach and the main results of this work.

# 1 Consistent Stochastic Utilities

We start by introducing the concept of a forward utility consistent with a given family of portfolios. All stochastic processes are defined on a filtered probability space $(\Omega, \mathcal{F}_{t\geq 0}, \mathbb{P})$ with complete filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. In general, $\mathcal{F}_0$ is assumed to be trivial. In the Itô framework considered afterwards, $(\mathcal{F}_t)_{t\geq 0}$ will be usual an augmented Brownian filtration.

## 1.1 Definition of Consistent Stochastic Utilities

A *progressive utility* $U$ is a *positive* adapted continuous random field $U(t,x)$, such that $t \geq 0$, $x > 0 \mapsto U(t,x)$ is an increasing concave function (in short utility function).

Obviously, this very general definition has to be compelled to represent more realistically the individual preferences of an investor in a given financial market, possibly changing over time. The idea is to calibrate these utilities with regard to some convex class (in particular vector space) of wealth processes, denoted by $\mathscr{X}$, on which utilities may have more properties.

As classical utility function, a progressive stochastic utility measures the relative satisfaction of any portfolio and gives a selection criterion which allows to identify an optimal choice of investment at any time. In general, we will impose below the uniqueness of the optimal process, to be as close as possible to the usual expectations of investors. Furthermore, the satisfaction for the optimal choice is maximum and will be preserved at all futures times which explains the martingale property in the definition below. On the other hand if the strategy in $\mathscr{X}$ fails to be optimal then it is better not to make investment. The fact of making a bad investment choice can be seen as a loss, compared with what he could won if he had followed the optimal policy.

From this, we suppose that the utility of any strategy is a supermartingale and so the optimum represents the reference (benchmark) for the investor.
The class $\mathscr{X}$ is a test-class which only allows us to specify the stochastic utility. Once his utility is defined, an investor can then turn to a portfolio optimization problem on the general financial market to establish his optimal policy or to calculate indifference prices.
Now we are able to define the $\mathscr{X}$-consistent stochastic utility as follows.

**Definition 1.1** (**$\mathscr{X}$-consistent Utility).** *A $\mathscr{X}$-consistent stochastic utility process $U(t,x)$ is a positive progressive utility with the following properties:*

- **Consistency with the test-class:** *For any admissible wealth process $X \in \mathscr{X}$, $\mathbb{E}(U(t,X_t)) < +\infty$ and*
$$\mathbb{E}(U(t,X_t)/\mathcal{F}_s) \leq U(s,X_s), \ \forall s \leq t \quad .a.s.$$
- **Existence of optimal wealth:** *For any initial wealth $x > 0$, there exists an optimal wealth process $X^* \in \mathscr{X}$, such that $X^*_0 = x$, and $U(s,X^*_s) = \mathbb{E}(U(t,X^*_t)/\mathcal{F}_s)\ \forall s \leq t$.*

*In short for any admissible wealth $X \in \mathscr{X}$, $U(t,X_t)$ is a positive supermartingale and a martingale for the optimal-benchmark wealth $X^*$.*

Our definition of consistent dynamic utilities differs from the one introduced by Musiela and Zariphopoulou [26, 29, 27, 30, 32] or Barrier and al. [10] by the fact that we do not require that the wealth processes $X$ are discounted. This variation offers more options and allows us to study the invariance of the class of stochastic utilities by change of numéraire. In any case, there is no fixed horizon.
**Remark** A deterministic utility $u$ is a $\mathscr{X}$-consistent utility only when the test-portfolios are local martingales. In this case, the optimal strategy is to do nothing.

**The Market Model** In this paragraph, we follow the presentation of Karatzas and Shreve [19]. We consider a securities market which consists of $d+1$ assets, one riskless asset, with price $S^0$ given by $dS^0_t = S^0_t r_t dt$ and $d$ risky assets. We model the price of the $d$ risky assets as a locally bounded positive semimartingale $S^i, \ \ i = 1,\dots,d$ defined on the filtered probability space $(\Omega, \mathcal{F}_{t\geq 0}, \mathbb{P})$.
A (self−financing) portfolio is defined as a pair $(x,\phi)$, where the constant $x$ is the initial value of the portfolio and the column vector $\phi = (\phi^i)_{1\leq i\leq d}$ is a predictable $S$-integrable process

specifying the amount of each asset held in the portfolio. The value process, also called wealth process, $X^\phi = (X_t^\phi)_{t\geq 0}$ of such portfolio $\phi$ is given by

$$\frac{X_t^\phi}{S_t^0} = \frac{x}{S_0} + \int_0^t \frac{\phi_\alpha}{S_\alpha^0}.d(\frac{S_\alpha}{S_\alpha^0}), \quad t \geq 0. \tag{1}$$

Let us denote by $\mathbb{X}^+$ the set of non negative wealth processes. To facilitate the exposition we only consider wealth processes in $\mathbb{X}^+$. This naturally leads us to characterize portfolios by means of relative weights $\pi$ in place of the amounts $\phi$. The relation between these two notions is easy since $\phi_t = (\pi_t^1 X_t^\phi(x), .., \pi_t^d X_t^\phi(x))^T$, where the transpose operator is denoted by $^T$. The advantage of the second formulation is that the assumption of positive wealth is automatically satisfied, since the previous equation becomes with the notation $X^\pi$ in place of $X^\phi$,

$$\frac{dX_t^\pi}{X_t^\pi} = r_t dt + \pi_t.(\frac{dS_t}{S_t} - r_t \mathbf{1} dt), \quad t \geq 0 \tag{2}$$

where the $d$-dimensional vector denoted by $\mathbf{1}$ is such all components are equal to 1. Let us now recall that a probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an equivalent local martingale measure if, for any $X \in \mathbb{X}^+$, $\frac{X}{S^0}$ is a local martingale under $\mathbb{Q}$. To ensure the absence of arbitrage opportunities, we postulate that the family of equivalent local martingale measures is not empty, (see [8], [6] for a precise statement and references). We stress that no assumption concerning completeness is made and in particular, many equivalent martingale measures may exist.

**Itô's Market:** Let $W = (W_1, W_2, ..., W_n)^T$ be a $n$-standard Brownian motion ($n \geq d$), defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is the $\mathbb{P}$-augmented filtration generated by the Brownian motion $W$.
The risky asset prices are continuous Itô's semimartingales with the dynamics:

$$\frac{dS_t^i}{S_t^i} = b_t^i dt + \sigma_t^i.dW_t, \quad \text{for } 0 \leq i \leq d \tag{3}$$

where the inner scalar product is denoted by "$.$". The coefficient $b^i$ represents the appreciation rate by time unit of the asset $i$ and $\sigma^i$ its volatility vector in $\mathbb{R}^n$, considered as a $n \times 1$ matrix. Denote by $b$ the appreciation rate column vector $n \times 1$ $(b^i)_{i=1,\dots,d}$, and by $\sigma_t$ the volatility matrix $n \times d$ ($n$ lines $d$ columns), whose $i^{th}$ column is the vector $\sigma_t^i$ for $i = 1, \dots, d$. The processes $b$, $\sigma$ and $r$ are $\mathcal{F}_t$ non-anticipating processes and satisfy some minimal appropriate integrability conditions. Using vector and matrix notation, we have $dS_t = S_t\big(b_t dt + \sigma_t^T dW_t\big)$ Moreover,

equation (2) may be rewritten as, $dX_t^\pi = X_t^\pi\big[\big(r_t + \pi_t.(b_t - r_t)\mathbf{1}\big)dt + \sigma_t\pi_t.dW_t\big]$. As usual, the matrix $(\sigma\sigma^T)(t,\omega)$ is assumed to be **non singular**. This assumption is equivalent to suppose that, for any $i \in 1..d$, the asset $S^i$ can not be replicated by an admissible portfolio. The existence of an equivalent local martingale measure in this framework implies that the excess of return vector belongs to the range of volatility matrix: in other words, there exists a $\mathcal{F}$-progressively measurable process $\eta \in \mathbb{R}^n$ such that $b_t - r_t\mathbf{1} = \sigma_t^T\eta_t$. Additional integrability Assumptions are necessary to ensure that the exponential martingale generated by $\eta.W$ is the density of some probability measure.

We get that the dynamics of the portfolio becomes $dX_t^\pi = X_t^\pi\big[r_t dt + \sigma_t\pi_t.(dW_t + \eta_t dt)\big]$ The key role is played by the volatility vector $\sigma\pi$. For this and in order to facilitate the exposition, we denote it by $\kappa := \sigma\pi$. To fix the notation, we denote by $\mathcal{R}_t^\sigma \subset \mathbb{R}^n$ the range of $\sigma_t$, and by $\mathcal{R}_t^{\sigma,\perp}$ the orthogonal vector subspace. By assumption, $\kappa_t$ is required to lie at any time $t$ in $\mathcal{R}_t^\sigma$. Replacing $X^\pi$ by $X^\kappa$, the above equation becomes

$$dX_t^\kappa = X_t^\kappa\big[r_t dt + \kappa_t.(dW_t + \eta_t dt)\big], \ \kappa_t \in \mathcal{R}_t^\sigma. \tag{4}$$

Note that under market assumptions ($\sigma_t^T\sigma_t$ non singular) there exists a unique vector $\pi_t$ such that $\kappa_t = \sigma_t\pi_t$.

The following short notation will be used extensively. Let $\mathcal{R}^\sigma$ be a vector subspace of $\mathbb{R}^n$. For any $\alpha \in \mathbb{R}^n$, $\alpha^\sigma$ is the orthogonal projection of the vector $\alpha$ onto $\mathcal{R}^\sigma$ and $\alpha^\perp$ is the orthogonal projection onto $\mathcal{R}^{\sigma,\perp}$. To express the orthogonal projection onto $\mathcal{R}^\sigma$, Musiela and Zariphopoulou [32] use the generalized inverse of $\sigma$, known as the Moore-Penrose inverse $\sigma^+$, characterized by the following identities

$$\sigma\sigma^+ = (\sigma\sigma^+)^T, \ \ \sigma^+\sigma = (\sigma^+\sigma)^T, \ \ \sigma\sigma^+\sigma = \sigma, \ \ \sigma^+\sigma\sigma^+ = \sigma^+$$

Indeed $\sigma\sigma^+$ is the orthogonal projection matrix onto $\mathcal{R}^\sigma$, and $\alpha^\sigma = \sigma\sigma^+\alpha$.

**Minimal Risk premium** The market incompleteness is described through the family of risk premium $\eta$. Since for any $\kappa \in \mathcal{R}^\sigma$, $\kappa.\eta = \kappa.\eta^\sigma$, we assume throughout this paper and without further mention that $\eta = \eta^\sigma \in \mathcal{R}^\sigma$. $\eta^\sigma$ is often referred to as the minimal risk premium.

# 2 Stochastic Partial Differential Equation

In this section, under some additional regularity assumptions, we focus on the Hamilton-Jacobi-Bellman stochastic PDE satisfied by a $\mathscr{X}$-consistent stochastic utility using essentially Itô-Ventzel's formula and techniques of dynamic programming established and developed in the classical theory of utility maximization (see for example H. Pham [14]). Additional regularity assumptions are necessary to advance in the study. From now, $\mathscr{X}$-consistent stochastic utilities $U(t,x)$ $\big(U(0,.)=u(.)\big)$ are described as Itô's semimartingales with spatial parameter $x>0$; in other words, $U(t,x)$ is a continuous random field with dynamics,

$$dU(t,x)=\beta(t,x)dt+\gamma(t,x).dW_t, \tag{5}$$

where, as in Kunita [25], the local characteristics $(\beta,\gamma)$ of $U$ and are assumed to be progressively random fields with values in $\mathbb{R}$ and $\mathbb{R}^n$ respectively.
We are concerned with the properties of the utility of admissible wealth processes. Before that, we want to give precise definition of the progressive utility, assumed to be of class $\mathcal{C}^2$, its derivatives and their dynamics properties.

## 2.1 Regular stochastic flows and Itô-Ventzel's formula

**Regular Stochastic flows** There are several difficulties in the definition of semimartingales depending on a parameter, as explained in H. Kunita [25] and R.A. Carmona et al. [2], (see Appendix A).
First let us point out that in general equality (5) holds for any $t$ except for a null set $N_x$. Then the semimartingale $U$ is well defined for $(t,x)$ if $\omega\in(\cup_{x\in\mathbb{R}_+}N_x)^c$. However the exceptional set $(\cup_{x\in\mathbb{R}_+}N_x)$ may not be a null set since it is an uncountable union of null sets. However if we suppose that local characteristic $(\beta,\gamma)$ of $U$ are $\delta$-Hölder, for some $\delta>0$ (see appendix A), then according to H. Kunita [25] (Theorems 3.1.2 p.75) using Kolmogorov's criterion, $U(t,x)$ has a continuous modification for which (5) holds almost surely.
A detailed discussion about these difficulties and their consequences in terms of dynamic representation and differential rules are provided in H. Kunita [25] and R.A. Carmona et al. [2]. The main results are also recalled in Appendix A. Here we only give a self-contained definition of the regularity in the sense of Kunita [25]. In particular, albeit the process $U$ and its

local characteristics ($\beta$, $\gamma$) are differentiable it is not enough as is shown in H. Kunita [25], to get that the dynamics of the derivative $\frac{\partial}{\partial x}U(t,x)$ is the derivative term by term of that of $U$. Let $m$ be a non-negative integer, $\beta$ be a real function in $\mathcal{C}^m([0,+\infty[\times[0,+\infty[)$ and $\gamma$ be a $\mathcal{C}^m([0,+\infty[\times[0,+\infty[)$ vector. We define the following seminorms for any compact $K$,

$$||\beta||_{m:K}(t) = \sup_{x\in K}\frac{|\beta(t,x)|}{1+|x|} + \sum_{1\leq\alpha\leq m}\sup_{x\in K}|\partial_x^\alpha\beta(t,x)|.$$

$$||\gamma||_{m:K}^{\sim}(t) = \sup_{x,y\in K}\frac{|\gamma^T(t,x).\gamma(t,y)|}{(1+|x|)(1+|y|)} + \sum_{1\leq\alpha_1,\alpha_2\leq m}\sup_{x,y\in K}|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\gamma^T(t,x).\gamma(t,y)|$$

For simplicity if a random field $(G(t,x))_{t\geq 0,x\geq 0}$ is of class $\mathcal{C}^{0,2}([0,+\infty[\times[0,+\infty[)$ we use the notation $G_x$ for $\frac{\partial}{\partial x}G$ and $G_{xx}$ for $\frac{\partial^2}{\partial x^2}G$.

**Definition 2.1.** *Let $m\geq 2$ and $F$ be a random field with spatial parameter $x$ defined on $\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ with local characteristics $(\beta,\gamma)$,*

$$F(t,x) = F(0,x) + \int_0^t\beta(s,x)ds + \int_0^t\gamma(s,x).dW_s. \tag{6}$$

*According to Kunita [25], $F$ is said to be $\mathcal{C}^{(m)}$ regular in the sense of Kunita if*

- *$F$ is continuous on the time and of class $\mathcal{C}^m(\mathbb{R}_+)$ on $x$.*
- *$\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ and the $N$-dimensional vector $\gamma:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}^N$ are $\mathcal{F}$-adapted random field continuous on the time and are of class $\mathcal{C}^{m-1}(\mathbb{R}_+)$ on the spatial parameter $x$ such that $||\beta||_{m-1:K}(t)$ and $||\gamma||_{m-1:K}^{\sim}(t)$ are integrable with respect to $t$, for any compact $K\subset[0,+\infty[$.*

Now we turn to the differential rules of semimartingales with spatial parameter. For this some other notations are needed. Let $0<\delta\leq 1$ and $K$ a compact of $\mathbb{R}_+$. For some random fields $f(t,x)$ and $g(t,x,y)$ we set

$$||f||_{\delta:K} := \sup_{\substack{x,y\in K\\ x\neq y}}\frac{|\partial_x^\alpha f(x)-\partial_x^\alpha f(y)|}{|x-y|^\delta},\quad ||g||_{\delta:K} := \sup_{\substack{x,x',y,y'\in K\\ x\neq x',y\neq y'}}\frac{|g(x,y)-g(x',y)-g(x,y')+g(x',y')|}{|x-x'|^\delta|y-y'|^\delta}.$$

Using these notations and according to Kunita [25],(Theorem 3.3.3 p.95, recalled in Appendix A), we have the following differential rules.

**Theorem 2.1** (Differential Rules)**.** *Let $F$ be a random field of class $\mathcal{C}^{0,1}([0,+\infty[\times[0,+\infty[)$ such that its local characteristics $(\beta,\gamma)$ are of class $\mathcal{C}^{0,1}([0,+\infty[\times[0,+\infty[)$. Assume that the derivative $\beta_x$ and $\gamma_x$ are $\delta$-Hölder, with $0<\delta\leq 1$ such that for any compact $K$ of $\mathbb{R}_+$, $||\beta||_{\delta:K}(t)$ and $||a^\gamma||_{\delta:K}(t)$ are integrable with respect to $t$, with $a^\gamma(t,x,y):=\gamma(t,x)^T.\gamma(t,y)$. Then the derivative $F_x$ of $F$ with respect to the spatial parameter $x$ satisfies, almost surely,*

$$F_x(t,x)=F_x(0,x)+\int_0^t\beta_x(s,x)ds+\int_0^t\gamma_x(s,x).dW_s \tag{7}$$

*Furthermore, if $F$ is of class $\mathcal{C}^{(m)}$, $m\geq 2$ in the sense of Kunita then $F_x$ is of class $\mathcal{C}^{(m-1)}$ in the sense of Kunita with local characteristics $(\beta_x,\gamma_x)$ which are of class $\mathcal{C}^{0,m-2}([0,+\infty[\times[0,+\infty[)$.*

**Itô-Ventzel's formula** Now, we need to study the dynamics of $U(t,X_t^\kappa)$ ($X^\kappa$ is a wealth process). Itô-Ventzel's formula is a generalization of classical Itô's formula where the deterministic function is replaced by a stochastic process depending on a real or multivariate parameter. This enables us to carry out computations in a stochastically modulated dynamic framework.

**Theorem 2.2** (Itô-Ventzel's Formula)**.** *Consider a random field $F:[0,+\infty[\times[0,+\infty[\to\mathbb{R}$ which is of class $\mathcal{C}^{(2)}$ in the sense of Kunita,*

$$F(t,x)=F(0,x)+\int_0^t\beta(s,x)ds+\int_0^t\gamma(s,x).dW_s,\ a.s. \tag{8}$$

*Furthermore, let $X$ be a continuous semimartingale with decomposition*

$$X_t=X_0+\int_0^t\mu_s^X ds+\int_0^t\sigma_s^X.dW_s$$

*Then $(F(t,X_t))$ is also a continuous semimartingale with decomposition*

$$\begin{aligned}F(t,X_t) &= F(0,X_0)+\int_0^t\beta(s,X_s)ds+\int_0^t\gamma(s,X_s).dW_s\\ &+ \int_0^t F_x(s,X_s)dX_s+\frac{1}{2}\int_0^t F_{xx}(s,X_s)\langle dX_s\rangle\\ &+ \int_0^t\gamma_x(s,X_s).\sigma_s^X ds.\end{aligned}$$

Let us now comment the dynamics of $F(t,X_t)$. The first line of the right hand side of this dynamic corresponds to the dynamics of the process $(F(t,x))_{t\geq 0}$ taken on $(X_t)_{t\geq 0}$, where the

second one is none other than the classical Itô formula, and the last one represents a correction term which can be written $\gamma_x(s, X_s).\sigma_s^X = \langle dF_x(s, x), dX_s\rangle|_{x=X_s}$.
We refer to Kunita [25], (Theorem 3.3.1, p.92) for more details and the proof of this result. In the following example, we illustrate this formula from the classical Itô's formula.

**Example: Itô's Formula** Let $f(t, \theta, x)$ be a deterministic function $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^{1,2,2}$. Denote by $\nabla_\theta$ the gradient with respect to $\theta$ and by $\Delta_{\theta\theta}$ the Hessian matrix with respect to $\theta$ where $t$ and $x$ are fixed.
Let $\Theta \in \mathbb{R}^n$ be an Itô's semimartingale $d\Theta_t = \mu_t^\Theta dt + \Sigma_t^{\Theta,T} dW_t$, with the diffusion generator

$$L_t^\Theta = \frac{\partial}{\partial t} f(t, \Theta_t, x) + \mu_t^\Theta \nabla_\theta + \frac{1}{2} trace[\Sigma_t^{\Theta,T} \Sigma_t^\Theta \Delta_{\theta\theta}].$$

Denote by $F$ the stochastic random field $F(t, x) \stackrel{def}{=} f(t, \Theta_t, x)$. By the classical Itô's formula

$$dF(t, x) = L_t^\Theta f(t, \Theta_t, x)dt + \Sigma_t^\Theta \nabla_\theta f(t, \Theta_t, x).dW_t$$

such that $F$ is a stochastic random field with local characteristics $\beta^F$ and $\gamma^F$ given by

$$\beta^F(t, x) = L_t^\Theta f(t, \Theta_t, x), \quad \gamma^F(t, x) = \Sigma_t^\Theta \nabla_\theta f(t, \Theta_t, x).$$

Let now $X$ be another real continuous semimartingale $dX_t = \mu_t^X dt + \sigma_t^X.dW_t$ and $L^X$ its stochastic diffusion operator without the term on $\frac{\partial}{\partial t}$.
We now compute the dynamics of $F(t, X_t) := f(t, \Theta_t, X_t)$ by the classical Itô's formula applied to the vector $(\Theta_t, X_t)$, with diffusion generator $L^{\Theta,X}$, and compare it with the Itô-Ventzel's formula. We obtain

$$\begin{aligned} dF(t, X_t) &= L_t^{\Theta,X} f(t, \Theta_t, X_t)dt + \left(\Sigma_t^\Theta \nabla_\theta f(t, \Theta_t, X_t) + f_x(t, \Theta_t, X_t)\sigma_t^X\right).dW_t \\ &= \beta^F(t, X_t)dt + \gamma^F(t, X_t).dW_t + \gamma_x^F(t, X_t).\sigma_t^X dt + L_t^X f(t, \Theta_t, X_t)dt \\ &+ f_x(t, \Theta_t, X_t)\sigma_t^X.dW_t. \end{aligned}$$

Denoting $\Delta_{\theta,x} f := \nabla_\theta f_x$, $L_t^{\Theta,X} := L_t^\Theta(t, \theta, x) + L_t^X(t, \theta, x) + \sigma_t^X.\sigma_t^\Theta \Delta_{\theta,x} f(t, \theta, x)$, and then,

$$\gamma_x^F(t, X_t).\sigma_t^X dt = \Sigma_t^\Theta \nabla_\theta f_x(t, \Theta_t, X_t).\sigma_t^X dt = \sum_j f_{x,\theta_j} < d\Theta_t^j, dX_t > .$$

## 2.2 Stochastic PDE of $\mathscr{X}$-consistent Dynamic Utilities

Using the same ideas as in interest rate modeling when studying the dynamics of the forward rates, or in the stochastic volatility models to characterize the drift of the stochastic implied volatility, we show how the consistency property constraints the random fields $\beta(t,x)$ and $\gamma(t,x)$ in terms of the random field $U$, its derivatives and the market parameters $(r_t,\eta^\sigma_t)$.

**Lemma 2.3** (Drift Constraint)**.** *Let $U$ be a progressive utility of class $\mathcal{C}^{(2)}$ in the sense of Kunita with local characteristics $(\beta,\gamma)$ as in (5). Then, for any admissible portfolio $X^\kappa$,*

$$\begin{aligned}
dU(t,X^\kappa_t) &= \Big(U_x(t,X^\kappa_t)X^\kappa_t\,\kappa_t+\gamma(t,X^\kappa_t)\Big).dW_t\\
&+\Big(\beta(t,X^\kappa_t)+U_x(t,X^\kappa_t)r_tX^\kappa_t+\frac{1}{2}U_{xx}(t,X^\kappa_t)\mathcal{Q}(t,X^\kappa_t,\kappa_t)\Big)dt,\\
\textit{where }\mathcal{Q}(t,x,\kappa) &:= \|x\kappa\|^2+2x\kappa.\Big(\frac{U_x(t,x)\eta^\sigma_t+\gamma_x(t,x)}{U_{xx}(t,x)}\Big).
\end{aligned}$$

*Since $\kappa\in\mathcal{R}^\sigma$, $\mathcal{Q}(t,x,\kappa)$ is only depending on $\gamma^\sigma_x(t,x)$, the orthogonal projection of $\gamma_x(t,x)$ on $\mathcal{R}^\sigma_t$. Let $\mathcal{Q}^*(t,x)=\inf_{\kappa\in\mathcal{R}^\sigma}\mathcal{Q}(t,x,\kappa)$; the minimum of this quadratic form is achieved at the optimal policy $\kappa^*$ given by*

$$\begin{cases}
x\kappa^*_t(x) &= -\frac{1}{U_{xx}(t,x)}\big(U_x(t,x)\eta^\sigma_t+\gamma^\sigma_x(t,x)\big)\\
x^2\mathcal{Q}^*(t,x) &= -\frac{1}{U_{xx}(t,x)^2}||U_x(t,x)\eta^\sigma_t+\gamma^\sigma_x(t,x))||^2=-||x\kappa^*_t(x)||^2.
\end{cases}\tag{9}$$

**Remark 2.1.** *Similar results may be established when the volatility vector of wealth processes has an affine form $\bar\kappa_t=\kappa_t+\alpha_t,\ \kappa_t\in\mathcal{R}^\sigma_t,\ \alpha_t\in\mathbb{R}^N$ and the market risk premium is a general $\eta$. In this framework the quadratic form is unchanged, $\mathcal{Q}(t,x,\bar\kappa)=\|x\bar\kappa\|^2+2x\bar\kappa.\big(\frac{U_x(t,x)\eta_t+\gamma_x(t,x)}{U_{xx}(t,x)}\big)$ but the optimization program is related to affine constraints, $\bar\kappa_t-\alpha_t\in\mathcal{R}^\sigma_t$. Denoting $\bar{\mathcal{Q}}^*(t,x,\alpha):=$ $\inf_{\bar\kappa\in\mathcal{R}^\sigma+\alpha}\mathcal{Q}(t,x,\bar\kappa)$, the optimal policy is*

$$x\bar\kappa^*_t=x\alpha_t+\Big(\frac{U_x(t,x)\eta_t+\gamma_x(t,x)}{U_{xx}(t,x)}-x\alpha_t\Big)^\sigma=\frac{U_x(t,x)\eta^\sigma_t+\gamma^\sigma_x(t,x)}{U_{xx}(t,x)}+x\alpha^\perp_t\tag{10}$$

*and finally*

$$\bar{\mathcal{Q}}^*(t,x,\alpha)=||\frac{U_x(t,x)\eta^\perp_t+\gamma^\perp_x(t,x)}{U_{xx}(t,x)}-x\alpha^\perp_t||^2-||\frac{U_x(t,x)\eta_t+\gamma_x(t,x)}{U_{xx}(t,x)}||^2\tag{11}$$

*This identities will be used extensively in the study of the dual process and the transformation by change of numeraire.*

*Proof.* $(i)$ The first assertion is a direct consequence of Itô-Ventzel formula applied to the composite process $\big(U(t,X_t^\kappa)\big)_{t\geq 0}$, where $X_t^\kappa$ is an admissible wealth process with dynamics given by (4), $dX_t^\kappa = X_t^\kappa\big(r_t dt + \kappa_t.(dW_t + \eta_t^\sigma dt)\big)$.
$(ii)$ The minimization program of the quadratic form $\mathcal{Q}(t,x,\kappa)$ corresponds to the square of the distance between $-\frac{U_x(t,x)\eta_t^\sigma+\gamma_x(t,x)}{U_{xx}(t,x)}$ and the vector space $\mathcal{R}_t^\sigma$, that yields to equation (9). Then, the minimum is given by $\mathcal{Q}^*(t,x) = -||x\kappa_t^*(x)||^2$. □

This lemma suggests that the constraint on the drift $\beta$ implies the consistency condition. The idea of the next theorem is to reformulate this constraint as a natural candidate for $\beta$.

**Theorem 2.4** (Utility-SPDE)**.** *Let $U$ be a progressive utility of class $\mathcal{C}^{(2)}$ in the sense of Kunita with local characteristics $(\beta,\gamma)$. Let us introduce the risk tolerance coefficient $\alpha_t^U(t,x) = -\frac{U_x(t,x)}{U_{xx}(t,x)}$ and the utility risk premium $\eta^U(t,x) = \frac{\gamma_x(t,x)}{U_x(t,x)}$ with its two components $\eta^{U,\sigma}\in\mathcal{R}^\sigma$, $\eta^{U,\perp}\in\mathcal{R}^{\sigma,\perp}$.*

*The quadratic form $x^2\mathcal{Q}(t,x,\kappa) = \|x\kappa_t\|^2 - 2\alpha^U(t,x)(x\kappa_t).\big(\eta_t^\sigma + \eta_t^{U,\sigma}(x)\big)$ achieves its minimum at the optimal policy,*

$$x\kappa_t^*(x) = -\frac{1}{U_{xx}(t,x)}(U_x(t,x)\eta_t^\sigma + \gamma_x^\sigma(t,x)) = \alpha^U(t,x)\big(\eta_t^\sigma + \eta_t^{U,\sigma}(x)\big) \tag{12}$$

*a) Assume the drift constraint to be of Hamilton-Jacobi-Bellman nonlinear type*

$$\beta(t,x) = -U_x(t,x)r_t x + \frac{1}{2}U_{xx}(t,x)\|x\kappa_t^*(t,x)\|^2 \tag{13}$$

*Then the progressive utility is a solution of the following HJB-SPDE*

$$dU(t,x) = -U_x(t,x)\left[xr_t + \frac{1}{2}\alpha^U(t,x)\|\eta_t^\sigma + \eta_t^{U,\sigma}(x)\|^2\right]dt + \gamma(t,x).dW_t,$$

*and for any admissible wealth $X_t^\kappa$, the process $U(t,X_t^\kappa)$ is a supermartingale.*
*b) Furthermore, if $\kappa_t^*(x)$ is assumed to be sufficiently smooth so that for any initial wealth $x>0$, the equation*

$$dX_t^* = X_t^*\big[r_t dt + \kappa_t^*(X_t^*).(dW_t + \eta_t^\sigma dt)\big] \tag{14}$$

*has at least one positive solution $X^*$, then $U(t,X_t^*)$ is a local martingale.*
*c) Moreover, if this local martingale $\big(U(t,X_t^*)\big)_{t\geq 0}$ is a martingale, then the progressive utility $U$ is a $\mathscr{X}$-consistent stochastic utility with optimal wealth process $X^*$.*

This theorem proves that the pair consisting on the investment universe and the derivative with respect to $x$ of the volatility denoted by $\gamma_x$ describes completely the evolution of the stochastic utility $U$. The drift $\beta(t,x)$ may be interpreted in term of the risk aversion $\eta^U$ and the volatility $\gamma$ of the utility. The optimal policy $\kappa^*$ is the best combination between the market risk premium $\eta^\sigma$ and the utility risk premium $\eta_t^{U,\sigma}(x) = \frac{\gamma_x^\sigma}{U_x}$.
The assumption (13) on the drift $\beta$ is a sufficient condition under which the consistence with the investment universe of the second assertion of Definition 1.1 is satisfied. Nevertheless, additional assumptions are needed on the existence of the wealth process $X^*$ for which $\big(U(t,X_t^*)\big)_{t\geq 0}$ is a martingale. This explains the assumptions of the second part of the result.

The Utility-SPDE poses several challenges. It is a fully nonlinear and non elliptic SPDE; the latter is a direct consequence of the "forward in time" nature of the involved stochastic optimization problem, for which there is no maximum principle. Thus, existing results of existence, uniqueness and regularity of weak (viscosity) solutions are not directly applicable. An additional difficulty comes from the fact that the volatility coefficient may depend on higher order derivatives of $U$, in which case the SPDE can not be turned into a regular PDE with random coefficients by using the method of stochastic characteristics. To overcome these difficulties, we propose a new method based on a stochastic change of variable; this method, that we call "stochastic flow method", allows us to construct explicit solutions of this Utility-SPDE. This will be the subject of Section 4.

*Proof.* All assertions are simple consequences of the previous lemma, since by the assumption on $\beta(t,x)$, $\beta(t,x) + x\,U_x(t,x)\,r_t + \frac{x^2 U_{xx}(t,x)}{2}\mathcal{Q}(t,x,\kappa) \leq 0,\ a.s.\ \forall \kappa \in \mathcal{R}^\sigma$, with equality for $\kappa_t^*(x)$. Therefore, $U(t,X_t^\kappa)$ is a positive supermartingale for any admissible strategy, and if equation (14) has a solution $X^*$, then the process $\big(U(t,X_t^*)\big)_{t\geq 0}$ is a local martingale.
The additional assumption that $\big(U(t,X_t^*)\big)$ is a true martingale yields the characterization of the $U(.,x)$ as $\mathcal{X}$-consistent utility. □

**Example: Change of probability and numeraire in standard utility function**
In this example, we study the $\beta$-HJB constraint (13) of Theorem 2.4 in the case of a $\mathscr{X}$-consistent stochastic utilities obtained by combining a standard utility function $v$ with some positive processes $N$ and $Z$. The advantage here is that the drift $\beta$ and the volatility $\gamma$ of the utility are given explicitly from $v$, $N$ and $Z$.

Let $v$ an $\mathcal{C}^2$ utility function and let $N$ and $Z$ two positive processes satisfying

$$\frac{dN_t}{N_t} = \mu_t^N dt + \sigma_t^N.dW_t, \quad \frac{dZ_t}{Z_t} = \mu_t^Z dt + \sigma_t^Z.dW_t, \quad Z_0 = 1.$$

Define the strictly increasing and concave process (with respect to $x$) $U$ by $U(t,x) = Z_t v(x/N_t)$. Applying Itô's Lemma and using identities $U_x(t,x) = \frac{Z_t}{N_t} v_x(\frac{x}{N_t})$, $U_{xx}(t,x) = \frac{Z_t}{N_t^2} v_{xx}(\frac{x}{N_t})$, it is straightforward to check that, for $x > 0$, the process $(U(t,x))_{t\geq 0}$ satisfies $dU(t,x) = \beta(t,x)dt + \gamma(t,x).dW_t$ where the local characteristics $\beta$ and $\gamma$ are given by

$$\begin{aligned} \beta(t,x) &= U(t,x)\mu_t^Z + xU_x(t,x)(-\mu_t^N + ||\sigma_t^N||^2 - \sigma_t^N.\sigma_t^Z) + \frac{1}{2}x^2 U_{xx}(t,x)||\sigma_t^N||^2 \quad (15)\\ \gamma(t,x) &= U(t,x)\sigma_t^Z - xU_x(t,x)\sigma_t^N. \quad (16)\end{aligned}$$

Given that $U$ is a progressive utility, we are interested in establishing conditions on the triplet $(v, N, Z)$ for the drift $\beta$ satisfies the HJB constraint (13).

**Proposition 2.5.** *Let $v$ be an utility function.*

*(i) The process $U$ defined by $U(t,x) = Z_t v(x/N_t)$ is a consistent stochastic utility if $Z$ is a martingale, $ZX^\kappa/N, \kappa \in \mathcal{R}^\sigma$ are positive local martingales and $\sigma^N \in \mathcal{R}^\sigma$. In this case the optimal policy is given by $\kappa_t^* = \sigma_t^N$.*

*(ii) In the special case where $v$ is a power or an exponential utility, then Condition : "$Z$ is martingale, $ZX^\kappa/N$ is a local martingale for any $\kappa \in \mathcal{R}^\sigma$ " can be relaxed.*

*– If $v$ is a power utility with risk aversion $a \neq 0$: $v(x) = \frac{x^a}{a}$, it suffices that the parameters of $Z$ and $N$ satisfying,*

$$\frac{1}{a}\mu_t^Z + r_t - \mu_t^N + \sigma_t^N.\eta_t^\sigma - \sigma_t^{N,\perp}.\sigma_t^{Z,\perp} + \frac{1}{2(1-a)}\|\eta_t^\sigma - \sigma_t^{N,\sigma} + \sigma_t^{Z,\sigma}\|^2 + \frac{1+a}{2}\|\sigma_t^{N,\perp}\|^2 = 0,$$

*so that $U(t,x) = Z_t v(x/N_t)$ is a consistent stochastic utility.*

*– If $v$ is an exponential utility it suffices to take $Z$ and $N$ satisfy*

$$\mu^N = r + \sigma^N.\eta^\sigma, \quad \mu^Z = \frac{1}{2}\|\eta^\sigma - \sigma^{N,\sigma} + \sigma^{Z,\sigma}\|^2, \quad \sigma^N \in \mathcal{R}^\sigma$$

*so that $U(t,x) = Z_t v(x/N_t)$ is a consistent stochastic utility.*

This result gives sufficient conditions under which $U$, defined above, is an $\mathscr{X}$-consistent stochastic utility. Note also that this example generalizes the one in [33] in which case $u$ is an exponential utility and provides a simillar sufficient condition.

*Proof.* To facilitate the exposition, let us denote by $\hat{r} = r - \mu^N + \sigma^N.\eta^\sigma$, $\hat{\eta} = \eta^\sigma - \sigma^N$. The volatility vector $\gamma$ being given by equation (16), Lemma 2.3 gives the optimal policy $\kappa^*$

$$\kappa^*_t(x) = -\frac{1}{xU_{xx}}\Big( - xU_{xx}(t,x)\sigma^{N,\sigma}_t + U_x(t,x)(\hat{\eta}^\sigma_t + \sigma^{Z,\sigma}_t)\Big). \tag{17}$$

Then, the drift of the utility process $U$ satisfies the HJB constraint (13) if and only if,

$$U\mu^Z_t + U_x\hat{r}_t x - xU_x\sigma^{N,\perp}.(\hat{\eta}_t + \sigma^Z_t) - \frac{(U_x)^2}{2U_{xx}}\big\|\hat{\eta}^\sigma_t + \sigma^{Z,\sigma}_t\big\|^2 + \frac{x^2U_{xx}}{2}\big\|\sigma^{N,\perp}_t\big\|^2(t,x) = 0.$$

Using that $U(t,x) = Z_t v(x/N_t)$ and simplifying by $xv(x)Z$, it follows from the definition of $\hat{\eta}$ and $\hat{r}$ that $\forall t \geq 0, x > 0$

$$\frac{v}{xv_x}\mu^Z_t + r_t - \mu^N_t + \sigma^N_t.\eta^\sigma_t - \sigma^{N,\perp}_t.\sigma^{Z,\perp}_t - \frac{v_x}{2xv_{xx}}\|\hat{\eta}^\sigma_t + \sigma^{Z,\sigma}_t\|^2 + \big(1 + \frac{xv_{xx}}{2v_x}\big)\|\sigma^{N,\perp}_t\|^2(t,x) = 0. \tag{18}$$

**The case:** $v/xv_x$ and $v_x/xv_{xx}$ are proportional, in turn $v$ is a power or exponential utility.

- $v(x) = x^a/a$**.** Then equation (18) becomes, $\forall t \geq 0$

$$\frac{1}{a}\mu^Z_t + r_t - \mu^N_t + \sigma^N_t.\eta^\sigma_t - \sigma^{N,\perp}_t.\sigma^{Z,\perp}_t + \frac{1}{2(1-a)}\|\eta^\sigma_t - \sigma^{N,\sigma}_t + \sigma^{Z,\sigma}_t\|^2 + \frac{1+a}{2}\|\sigma^{N,\perp}_t\|^2 = 0.$$

- $v(x) = -\frac{1}{c}e^{-cx}$, $c > 0$. Then $\forall t \geq 0$, $x > 0$,

$$\mu^Z - \frac{1}{2}\|\eta^\sigma_t - \sigma^{N,\sigma}_t + \sigma^{Z,\sigma}_t\|^2 - cx\Big(r_t - \mu^N_t + \sigma^N_t.\eta^\sigma_t - \sigma^{N,\perp}_t.\sigma^{Z,\perp}_t\Big) + \big(\frac{cx^2}{2} - cx\big)\|\sigma^{N,\perp}_t\|^2 = 0.$$

Obviously, this is a second order polynomial identically null, consequently all coefficients are nulls, i.e., $\hat{r} = r - \mu^N + \sigma^N.\eta^\sigma = 0$, $\mu^Z = \frac{1}{2}\|\eta^\sigma - \sigma^{N,\sigma} + \sigma^{Z,\sigma}\|^2$, $\sigma \in \mathcal{R}^\sigma$

**Second case :** $v/xv'$ and $v'/xv''$ are not proportional, then it is immediate that all terms of (18) are equal to zero, in turn $\tilde{r} = 0$, $\mu^Z = 0$, $\sigma \in \mathcal{R}^\sigma$, $\eta^\sigma - \sigma^N + \sigma^Z \in \mathcal{R}^{\sigma,\perp}$ and hence the optimal strategy $\kappa^*$ in (17) is simply $\sigma^N$.

To summarize the situation: $Z$ is a martingale, $X^\kappa/N$ is a martingale under the probability $\mathbb{Q}^Z$ defined by $d\mathbb{Q}^Z/d\mathbb{P} = Z$ and $\sigma^N \in \mathcal{R}^\sigma$.

□

As in the classical theory of optimal choice of portfolio in expected utility framework ([24], [40]), the process $\big(U_x(t, X^*_t)\big)$ has nice properties and a central place in the dual problem introduced in the next section. For notational simplicity in the next result we do not recall the dependence of the optimal wealth relative to its intial condition $x$ contrary to Section 4 where it plays a very important role..

**Proposition 2.6.** *Let $U$ be a progressive utility of class $\mathcal{C}^{(3)}$ in the sense of Kunita, with local characteristics $(\beta,\gamma)$. Assume that all assumptions of Theorem 2.4 hold true, in particular that $X^*$ is a solution of $dX^*_t = X^*_t\big[r_t dt + \kappa^*_t(X^*_t).(dW_t + \eta^\sigma_t dt)\big]$.*

*Let $L^*$ be the stochastic diffusion operator of $X^*$, $L^*_{t,x} = \frac{1}{2}\|x\kappa^*_t(x)\|^2\frac{\partial^2}{\partial x^2} + \{r_t x + (x\kappa^*_t(x)).\eta^\sigma_t\}\frac{\partial}{\partial x}$.*
*i) Then, $U_x$ is of class $\mathcal{C}^{(2)}$ in the sense of Kunita with local characteristics $(\beta_x,\gamma_x)$ and*

$$\begin{cases} \gamma_x(t,x) + U_{xx}(t,x)(x\kappa^*_t(x)) = -U_x(t,x)\eta^\sigma_t + \gamma^\perp_x(t,x) \\ \beta_x(t,x) = -U_x(t,x)r_t - L^*_{t,x}U_x(t,x) - (x\kappa^*_t(x)).\gamma_{xx}(t,x) \end{cases}$$

*ii) The semimartingale $U_x(t,X^*_t)$ has the following dynamics*

$$dU_x(t,X^*_t) = U_x(t,X^*_t)\big[-r_t dt + \big(\eta^{U,\perp}_t(X^*_t) - \eta^\sigma_t\big)\big).dW_t\big] \tag{19}$$

*In particular, for any admissible wealth process $X^\kappa$ ($\kappa \in \mathcal{R}^\sigma$), $(X^\kappa_t\, U_x(t,X^*_t))$ is a local martingale and a martingale if $X^\kappa = X^*$.*

This result shows that $U_x(t,X^*_t)$ plays the role of a state price density process, defined in Definition 3.1.

*Proof.* Theorem 2.1 shows that $U_x$ is of class $\mathcal{C}^{(2)}$ in the sense of Kunita, with local characteristics $(\beta_x,\gamma_x)$. On the other hand, by Theorem 2.4, we have the identities $\beta(t,x) = -xU_x(t,x)\,r_t + \frac{1}{2}x^2U_{xx}(t,x)\,||\kappa^*_t(x)||^2$ and $U_{xx}(t,x)(x\kappa^*_t(x)) = -\big(U_x(t,x)\eta^\sigma_t + \gamma^\sigma_x(t,x)\big)$. This second identity is useful to calculate $\frac{1}{2}U_{xx}(t,x)\partial_x(\,||x\kappa^*_t(x)||^2) = U_{xx}(t,x)\big((x\kappa^*_t(x)).\partial_x(x\kappa^*_t(x))\big)$. Taking the derivative with respect to $x$ in this second identity, it follows that

$$U_{xxx}(t,x)(x\kappa^*_t(x)) + U_{xx}(t,x)\partial_x(x\kappa^*_t(x)) = -\big(U_{xx}(t,x)\eta^\sigma_t + \gamma^\sigma_{xx}(t,x)\big).$$

In fact we are interested in the inner product with the vector $x\kappa^*_t(x)$ that yields the following equality written in an appropriate form

$$\begin{aligned} &\frac{1}{2}U_{xxx}(t,x)||x\kappa^*_t(x)||^2 + U_{xx}(t,x)\big((x\kappa^*_t(x)).\partial_x(x\kappa^*_t(x))\big) \\ &= -\big\{\frac{1}{2}U_{xxx}(t,x)||x\kappa^*_t(x)||^2 + U_{xx}(t,x)\eta^\sigma_t.(x\kappa^*_t(x))\big\} - \gamma^\sigma_{xx}(t,x).(x\kappa^*_t(x)). \end{aligned}$$

It is easy to recognize the first line as the derivative of $\frac{1}{2}U_{xx}(t,x)\,||x\kappa^*_t(x)||^2$ and the second line as related to the diffusion operator $L^*_{t,x}$. In this form the relation $\beta_x(t,x) = -U_x(t,x)r_t -$

$L^* U_x(t,x) - (x\kappa^*_t(x)).\gamma_{xx}(t,x)$ is easy to establish.
We have now all the elements needed to calculate the dynamics of $U_x(t,X^*_t)$ using Itô-Ventzel formula

$$
\begin{aligned}
dU_x(t,X^*_t) &= \big(\gamma_x(t,X^*_t) + U_{xx}(t,X^*_t)X^*_t\kappa^*_t(X^*_t)\big).dW_t \\
&+ \big\{\beta_x(t,X^*_t) + L^*U_x(t,X^*_t) + \gamma_{xx}(t,X^*_t).(X^*_t\kappa^*_t(X^*_t))\big\}dt.
\end{aligned}
$$

Note that in the last inner product, we can replace $\gamma_{xx}(t,X^*_t)$ by its orthogonal projection $\gamma^\sigma_{xx}(t,X^*_t)$ on the space $\mathcal{R}^\sigma$. Thanks to the previous calculation, the expression in the brackets of the second line is exactly $-r_tU_x(t,X^*_t)$. The diffusion coefficient may also be simplified into $-U_x(t,X^*_t)\eta^\sigma_t + \gamma^\perp_x(t,X^*_t)$. So, we obtain the remarkably simple dynamics of $U_x(t,X^*_t)$

$$
dU_x(t,X^*_t) = U_x(t,X^*_t)\big[-r_tdt + (\eta^{U,\perp}_t(X^*_t) - \eta^\sigma_t).dW_t\big].
$$

Let us now prove the last sentence. Given an admissible wealth process, $dX^\kappa_t = X^\kappa_t\big(r_tdt + \kappa_t(dW_t + \eta^\sigma_t dt)\big)$, standard Itô's calculus provides an explicit form for the dynamics of $Z^\kappa_t = X^\kappa_t U_x(t,X^*_t)$ as

$$
\begin{aligned}
\frac{dZ^\kappa_t}{Z^\kappa_t} &= \frac{dX^\kappa_t}{X^\kappa_t} + \frac{dU_x(t,X^*_t)}{U_x(t,X^*_t)} + < \frac{dX^\kappa_t}{X^\kappa_t}, \frac{dU_x(t,X^*_t)}{U_x(t,X^*_t)} > \\
&= \big[\kappa_t - \eta^\sigma_t + \eta^{U,\perp}_t(X^*_t)\big].dW_t,
\end{aligned}
$$

which implies that $Z^\kappa_t = X^\kappa_t U_x(t,X^*_t)$ is a local martingale for any $\kappa \in \mathcal{R}^\sigma$. In particular the volatility coefficient of $Z^*$ is $\sigma^{Z,*}_t = (\kappa^*_t(X^*_t) - \eta^\sigma_t + \eta^{U,\perp}_t(X^*_t))$. To show that the positive local martingale $Z^*$ is a martingale, we use the concavity of the utility $U$, with the fact that $U(0) = 0$. As consequence, $Z^*_t = X^*_t U_x(t,X^*_t) \leq U(t,X^*_t)$; since by assumption $U(.,X^*_.)$ is a martingale, the same property is true for $Z^*$. This completes the proof. $\square$

# 3 Duality

After having introduced the consistent stochastic utilities and established the associated SPDEs, several questions remain open at this stage. Indeed, we have shown that the volatility $\gamma$ of these utilities plays a fundamental role since it completely describes the stochastic dynamics of utilities and the optimal policy. It now remains to give an interpretation of the orthogonal part

$\gamma_x^{\perp}$. The classical theory leads naturally to introduce the convex conjugate function $\tilde{U}(t,y) \stackrel{def}{=} \inf_{x>0, x\in Q^+}\big(U(t,x) - x\,y\big)$ (also called the Legendre-Fenchel transform) of $U(t,x)$.

We want to show that this conjugate random field is a solution of a dual utility SPDE, consistent with a convex family $\mathcal{Y}$ of semimartingales called density processes parametrised by their so-called risk premium $\nu^{\perp}$. Then the optimal risk premium $\nu^{\perp,*}$ is related to $\eta^{U,\perp} = \gamma_x^{\perp}/U_x$.

In the classical theory of concave functions $f$ and their conjugates $\tilde{f}$, the monotone functions $f_x$ and $-\tilde{f}_y$ are inverse of each other, i.e. $-\tilde{f}_y(y) = f_x^{-1}(y)$; in the stochastic framework, monotone functions are replaced by stochastic monotone flows and their inverse flows whose dynamics are given by the Itô-Ventzel formula. For simplicity, we present these results separately and in an appropriate form.

## 3.1 Local characteristics of inverse flows

Let $\phi$ and $\psi$ be two one-dimensional stochastic flows, with dynamics

$$\begin{aligned} d\phi(t,x) &= \mu^{\phi}(t,x)dt + \sigma^{\phi}(t,x).dW_t, \\ d\psi(t,x) &= \mu^{\psi}(t,x)dt + \sigma^{\psi}(t,x).dW_t. \end{aligned}$$

From Itô-Ventzel's formula, under regularity assumptions, the compound random field $\phi\circ\psi(t,x) = \phi(t,\psi(t,x))$ is a semimartingale whose characteristics are given explicitly from those of $\phi$ and $\psi$ and their derivatives.

**Theorem 3.1.** *Suppose that $\phi$ is a $\mathcal{C}^{(2)}$ regular random field in the sense of Kunita, and $\psi(t,x)$ is a continuous semimartingale. Then the random field $\phi\circ\psi(t,x) = \phi(t,\psi(t,x))$ is a continuous semimartingale with decomposition*

$$\begin{aligned} d(\phi\circ\psi)(t,x) &= \mu^{\phi}\big(t,\psi(t,x)\big)dt + \sigma^{\phi}\big(t,\psi(t,x)\big).dW_t \\ &+ \phi_x\big(t,\psi(t,x)\big)d\psi(t,x) + \frac{1}{2}\phi_{xx}\big(t,\psi(t,x)\big)||\sigma^{\psi}(t,x)||^2 dt \\ &+ \sigma_x^{\phi}\big(t,\psi(t,x)\big).\sigma^{\psi}(t,x)dt. \end{aligned} \tag{20}$$

*The volatility of the compound process $\phi\circ\psi$ is given by*

$$\sigma^{\phi\circ\psi}(t,x) = \sigma^{\phi}\big(t,\psi(t,x)\big) + \phi_x\big(t,\psi(t,x)\big)\sigma^{\psi}(t,x).$$

The next proposition, used several times throughout this paper, gives the decomposition of the inverse of a strictly monotone stochastic flow.

**Proposition 3.2** (Inverse flow dynamics)**.** *Let $\phi$ be a strictly monotone flow, regular in the sense of Kunita, with characteristics $(\mu^\phi(t,x),\sigma^\phi(t,x))$. The inverse process $\xi$ of $\phi$ is defined on the range of $\phi$ by $\phi(t,\xi(t,y))=y$.*

*i) The inverse flow $\xi(t,y)$ has a dynamics given in terms of the old variables by:*

$$d\xi(t,y)=-\xi_y(t,y)\big(\mu^\phi(t,\xi)dt+\sigma^\phi(t,\xi).dW_t\big)+\frac{1}{2}\partial_y\Big(\xi_y(t,y)\|\sigma^\phi(t,\xi)\|^2\Big)dt$$

*ii) With the new variables, using that $\sigma^\xi(t,y)=-\xi_y(t,y)\sigma^\phi(t,\xi(t,y))$*

$$d\xi(t,y)=\sigma^\xi(t,y).dW_t+\Big(\frac{1}{2}\partial_y\big(\frac{\|\sigma^\xi(t,y)\|^2}{\xi_y(t,y)}\big)-\mu^\phi(t,\xi(t,y))\xi_y(t,y)\Big)dt$$

It is interesting to observe that the local characteristics of the inverse flow $\xi$ can be easily interpreted as some derivatives. This point will play a crucial role in the sequel. The mathematical formulation of this remark is given in the following corollary, where the assumptions of Proposition 3.2 are made.

**Corollary 3.3.** *Let $(\Phi(t,x),M^\phi(t,x),\Sigma^\phi(t,x))$ be the primitives, null at $x=0$, of $\phi(t,x)$, $(\mu^\phi(t,x),\sigma^\phi(t,x))$ respectively, so that the $\Phi(t,x)$ dynamics is $d\Phi(t,x)=M^\phi(t,x)dt+\Sigma^\phi(t,x).dW_t$. Then, the dynamics of the random field $\Xi(t,y)=\int_y^{+\infty}\xi(t,z)dz$ is*

- *In old variables,*

$$d\Xi(t,y)=\Sigma^\phi\big(t,\xi(t,y)\big).dW_t+M^\phi\big(t,\xi(t,y)\big)dt+\frac{1}{2}\Xi_{yy}(t,y)\|\Sigma^\phi_x\big(t,\xi(t,y)\big)\|^2dt$$

- *In new variables, with the notations $M^\xi(t,y)=M^\phi(t,-\Xi_y(t,y))$ and $\Sigma^\xi(t,y)=-\Sigma^\phi(t,-\Xi_y(t,y))$,*

$$d\Xi(t,y)=\Sigma^\xi(t,y).dW_t+\Big(M^\xi(t,y)+\frac{1}{2}\frac{\|\Sigma^\xi_y(t,y)\|^2}{\Xi_{yy}(t,y)}\Big)dt. \tag{21}$$

**Remark .** *Note that if the process $\phi$ is strictly increasing on $x$ then its primitive $\Phi$ is a progressive utility (concave increasing random field) with a convex conjugate random field $\Xi$ satisfying* (21)*. Moreover, for a particular choice of the drift $M^\Phi$ (HJB type), $\Phi$ is a consistent utility and the dynamics* (21) *gives the dual SPDE.*

*Proof.* The proof of Proposition 3.2 is essentially based on the generalized Itô's formula established in the Appendix. For simplicity, we denote by $(\mu^{\xi}, \sigma^{\xi})$ the local characteristic of $\xi$ assumed to be regular. By Itô-Ventzel's formula, we have

$$\begin{aligned} d\phi(t,\xi(t,y)) &= 0 \\ &= \mu^{\phi}(t,\xi(t,y))dt + \sigma^{\phi}(t,\xi(t,y)).dW_t + \phi_x(t,\xi(t,y))d\xi(t,y) \\ &+ \frac{1}{2}\phi_{xx}(t,\xi(t,y)) < d\xi(t,y) > +\sigma^{\phi}_x(t,\xi(t,y)).\sigma^{\xi}(t,y)dt \end{aligned}$$

Recalling the following identities

$$\phi_x(t,\xi(t,y)) = \frac{1}{\xi_y(t,y)}, \quad \phi_{xx}(t,\xi(t,y)) = -\frac{\xi_{yy}(t,y)}{(\xi_y(t,y))^3},$$

we can express the parameters of the decomposition in terms of $\xi, \xi_y$, and $\xi_{yy}$ and the diffusion coefficient $\sigma^{\xi}(t,y)$ of $\xi$, since $\sigma^{\xi}(t,y) = -\xi_y(t,y)\sigma^{\phi}(t,\xi(t,y))$. It is immediate that

$$\mu^{\xi}(t,y) = -\xi_y(t,y)\mu^{\phi}(t,\xi(t,y)) + \xi_y(t,y) < \sigma^{\phi}(t,\xi(t,y)), \sigma^{\phi}_y(t,\xi(t,y)) > +\frac{1}{2}\xi_{yy}(t,y)\|\sigma^{\phi}(t,\xi(t,y))\|^2$$

In terms of the stochastic random fields $\mu^{\phi}$ and $\sigma^{\phi}$, this may be written as

$$\mu^{\xi}(t,y) = -\xi_y(t,y)\mu^{\phi}(t,\xi(t,y)) + \frac{1}{2}\partial_y\big[\xi_y(t,y)\|\sigma^{\phi}(t,\xi(t,y))\|^2\big].$$

In terms of their own parameters, it follows from the strict monotonicity of $\xi$ that

$$\mu^{\xi}(t,y) = -\xi_y(t,y)\mu^{\phi}(t,\xi(t,y)) + \frac{1}{2}\partial_y\big[\|\sigma^{\xi}(t,y)\|^2/\xi_y(t,y)\big].$$

The proof of Proposition 3.2 is now complete.
The proof of Corollary 3.3 is achieved, first by reconciling the results of the previous proposition and the following identities,

$$(\Phi_x)^{-1}(t,y) = -\Xi_y(t,y), \; \Phi_{xx}(t,-\Xi_y(t,y)) = -\frac{1}{\Xi_{yy}(t,y)}, \; \text{ and } -C_x(t,-\Xi_y(t,y)) = \frac{D_y(t,y)}{\Xi_{yy}(t,y)}$$

and second by integrating with respect to $y$, using the initial condition $\Xi(t,0) = 0$. □

## 3.2 Convex conjugate of consistent stochastic utility and dual utility SPDE's

In this paragraph, properties and dynamics of the convex conjugate $\tilde{U}$ of a consistent stochastic utility $U$ are investigated. In particular we show that if the drift $\beta$ of the utility $U$ is of HJB type

(equation (13)), then the drift of the convex conjugate $\tilde{U}$ denoted by $\tilde{\beta}$ is also of HJB type and conversely. This property implies that the random field $\tilde{U}$ is a progressive convex conjugate random field, consistent with a familly of state price density processes, introduced below. We apply the results of the previous paragraph to the stochastic flow $\big(U_x(t,x)\big)$ and its inverse $\big(-\tilde{U}_y(t,y)\big)$.

The formula of inverse flows yields easily the dynamics of $\tilde{U}(t,y)$ from the dynamics of $U(t,x)$. Based on Proposition 3.2 and Corollary 3.3, we derive a stochastic partial differential equation whose convex conjugate process $\tilde{U}(t,y)$ is a solution.

**Proposition 3.4.** *Let $U$ be a consistent progressive utility of class $\mathcal{C}^{(3)}$ in the sense of Kunita, with risk tolerance $\alpha_t^U(t,x) = -\frac{U_x(t,x)}{U_{xx}(t,x)}$ and utility risk premium $\eta^U(t,x) = \frac{\gamma_x(t,x)}{U_x(t,x)}$. Assume that $U$ satisfies the utility SPDE with the $\beta$ constraint (13). Let $\tilde{U}$ be its dual convex conjugate, null if $y = +\infty$. Then,*

*(i-a) In old variables, $\tilde{U}$ satisfies $d\tilde{U}(t,y) = \beta^1(t,-\tilde{U}_y(t,y))dt + \gamma(t,-\tilde{U}_y(t,y)).dW_t$, where*

$$\begin{cases} \beta^1(t,x) = \beta(t,x) - \dfrac{1}{2U_{xx}(t,x)}\|\gamma_x(t,x)\|^2 \\ \qquad\qquad = U_x(t,x)\Big( - xr_t - \dfrac{\alpha^U(t,x)}{2}\big(||\eta_t^{U,\sigma}(x) + \eta_t^\sigma||^2 - ||\eta_t^U(x)||^2\big)\Big) \end{cases} \tag{22}$$

*(i-b) In new variables,*

$$\begin{cases} \tilde{\beta}(t,y) = \beta^1(t,-\tilde{U}_y(t,y)),\ \ \tilde{\gamma}(t,y) = \gamma(t,-\tilde{U}_y(t,y)),\ \ \tilde{\gamma}_y(t,y) = -y\tilde{U}_{yy}(t,y)\eta^U(t,-\tilde{U}_y(t,y)) \\ \tilde{\beta}(t,y) = y\tilde{U}_y(t,y)r_t + \dfrac{1}{2\tilde{U}_{yy}(t,y)}\big(\|\tilde{\gamma}_y(t,y)\|^2 - \|\tilde{\gamma}_y^\sigma(t,y) + y\tilde{U}_{yy}(t,y)\eta_t^\sigma\|^2\big) \end{cases}$$

*(ii) Optimization Programs: The drift $\beta^1(t,x)$ and $\tilde{\beta}(t,y)$ are the value of two optimization programs achieved, respectively, at the optimal policies $\theta^*(t,x) = \eta_t^{U,\perp}(x)$ and $\nu_t^*(y) = \theta^*(t,-\tilde{U}(t,y)) = -\tilde{\gamma}_y^\perp(t,y)/y\tilde{U}_{yy}(t,y)$.*

$$\begin{cases} \beta^1(t,x) = U_x(t,x)\Big[ - xr_t - \dfrac{1}{2}\alpha^U(t,x)\big(\inf_{\theta\in\mathcal{R}^{\sigma,\perp}} ||\theta_t - \big(\eta_t^\sigma + \eta_t^U(x)\big)||^2 - \|\eta_t^U(x)\|^2\big)\Big] \\ \tilde{\beta}(t,y) = y\tilde{U}_y(t,y)r_t - \dfrac{1}{2}y^2\tilde{U}_{yy}(t,y)\inf_{\nu_t\in\mathcal{R}^{\sigma,\perp}}\{||\nu_t - \eta_t^\sigma||^2 + 2\big(\nu_t - \eta_t^\sigma\big).\big(\dfrac{\tilde{\gamma}_y(t,y)}{y\tilde{U}_{yy}(t,y)}\big)\} \end{cases} \tag{23}$$

First, observe that as $-\tilde{U}_y$ is the inverse flow of $U_x$, the dynamic of the convex conjugate $\tilde{U}$ of $U$ becomes a simple consequence of Corollary 3.3. Second, the orthogonal part of the utility risk premium $\eta^{U,\perp} := \gamma_x^{\perp}/U_x$ is the optimal policy of the dual problem in $(ii)$. Third, given that $\beta$ is associated with an optimization program the dual drift $\tilde{\beta}$, it is also constrained by a HJB type relation in the new variables, which means that the convex conjugate $\tilde{U}$ is consistent with some given family of the state density processes.

*Proof.* By regularity assumptions, using Theorem 2.1, $U_x(t,x), \beta_x(t,x)$ and $\gamma_x(t,x)$ are regular enough to apply Itô-Ventzel formula. The assumptions of Proposition 3.2 and Corollary 3.3 are satisfied and hence the dynamics of the convex conjugate is a direct consequence of Corollary 3.3. Let us now recall that the drift $\beta(t,x)$ of $U(t,x)$ is given in Theorem 2.4 by

$$\beta(t,x) = U_x(t,x)\big(-xr_t - \frac{\alpha^U(t,x)}{2}||\eta_t^{U,\sigma}(x) + \eta_t^\sigma||^2\big).$$

Combining this identity with Definitions of the random fields $\alpha^U$ and $\eta^U$ yields (22). In other hand, the formula for $\beta^1(t,x)$ in (23) is a consequence of the following property of the orthogonal projection: the norm of the projection on $\mathcal{R}^\sigma$ is the distance to the orthogonal vector space $\mathcal{R}^{\sigma,\perp}$. So, for any vector $a \in R^N$, $||a^\sigma||^2 = \inf_{\nu\in\mathcal{R}^{\sigma,\perp}}||\nu - a||^2$. Replacing $a$ by $\big(\eta_t^\sigma + \eta_t^{U,\sigma}(x)\big)$ yields the result.

Now, we focus on the drift $\tilde{\beta}$ of $\tilde{U}(t,y)$ in new variables, using essentially the following identities,

$$\frac{U_x^2(t,-\tilde{U}_y(t,y))}{2U_{xx}(t,-\tilde{U}_y(t,y))} = \frac{1}{2}y^2\tilde{U}_{yy}(t,y), \quad \frac{\tilde{\gamma}_y(t,y)}{\tilde{U}_{yy}(t,y)} = -\gamma_x(t,-\tilde{U}_y(t,y)), \quad \frac{\tilde{\gamma}_y(t,y)}{y\tilde{U}_{yy}(t,y)} = -\eta_t^U(-\tilde{U}_y(t,y))$$

We get the desired formula for $\tilde{\beta}$ in (23).

$$\tilde{\beta}(t,y) = y\tilde{U}_y(t,y)r_t - \frac{1}{2}y^2\tilde{U}_{yy}(t,y)\inf_{\nu_t\in\mathcal{R}^{\sigma,\perp}}\{||\nu_t + \eta_t^\sigma||^2 + 2\big(\nu_t - \eta_t^\sigma\big).\big(\frac{\tilde{\gamma}_y(t,y)}{y\tilde{U}_{yy}(t,y)}\big)\}$$

On the other hand by orthogonal projection on $\mathcal{R}_t^{\sigma,\perp}$ and using the fact that $\eta_t^\sigma \in \mathcal{R}_t^\sigma$, there exists one and only one optimal process $\nu^*$ given by

$$\nu_t^*(y) = \frac{-\tilde{\gamma}_y^{\perp}(t,y)}{y\tilde{U}_{yy}(t,y)} = \eta_t^{U,\perp}(-\tilde{U}_y(t,y))$$

By the same argument we get $\theta^*(t,x) = \eta_t^{U,\perp}(x)$. Which achieves the proof. □

Let us now focus on the dual optimization problem.

**Definition 3.1** (State price density process)**.** *A Itô semimartingale $Y^\nu$ is called a state price density process if for any wealth process $X^\kappa$, $\kappa \in \mathcal{R}^\sigma$, $Y^\nu X^\kappa$ is a local martingale. It follows that $Y^\nu$ satisfies,*

$$\frac{dY_t^\nu}{Y_t^\nu} = -r_t dt + (\nu_t - \eta_t^\sigma).dW_t, \quad \nu_t \in \mathcal{R}^{\sigma,\perp}. \tag{24}$$

*$\mathscr{Y}$ is the family of all state density processes $\mathscr{Y} := \{Y^\nu,\ \nu \in \mathcal{R}^{\sigma,\perp},\ \ Y^\nu$ satisfies (24)$\}$*

Obviously the class $\mathscr{Y}$ is not empty, since taking $\nu \equiv 0$, $Y^0$ is the classical minimal density process where the pricing of future cash-flow at time $T$ is obtained by first discounting between $t$ and $T$ the cash value at $T$ with the short rate $r_t$, and then by taking the conditional expected value with respect to the minimal martingale measure. Moreover, any state density process $Y^\nu$ is the product of $Y^0$ by the density martingale $L_t^\nu = \exp\big(\int_0^t \nu_s.dW_s - 1/2\int_0^t |\nu_s|^2 ds\big)$.

We obtain an interesting interpretation of the volatility risk premium in terms of optimal density process.

Let now, turn to the main result of this section which is based on the interpretation of the drift $\tilde\beta$ as an optimization program.

**Theorem 3.5.** *Let $U$ be a consistent progressive utility of class $\mathcal{C}^{(3)}$, in the sense of Kunita, satisfying the $\beta$ HJB constraint. Then, its conjugate process $\tilde U(t,y)$ (convex decreasing stochastic flow) is consistent with the family of state density processes $\mathscr{Y}$, in the following sense:*
*$\tilde U(t, Y^\nu)$ is a submartingale for any $Y^\nu \in \mathscr{Y}$, and a martingale for some process $Y^{\nu^*} (:= Y^*)$.*
*The optimal process can be chosen as $Y_t^*(y) = U_x\big(t, X_t^*(-\tilde u_y(y))\big)$, whose dynamics is*

$$\frac{dY_t^*}{Y_t^*} = -r_t dt + \big(\nu_t^*(Y_t^*) - \eta_t^\sigma\big).dW_t.$$

*where the dual optimal parameter $\nu_t^*(y)$ is given by*

$$\nu_t^*(y) = \frac{-\tilde\gamma_y^\perp(t,y)}{y\tilde U_{yy}(t,y)} = \frac{\gamma_x^\perp\big(t, -\tilde U_y(t,y)\big)}{y} = \eta_t^{U,\perp}(-\tilde U_y(t,y)).$$

**Remark .** *Let $\mathcal{Y}(t,x) := U_x(t, X_t^*(x))$, if $X_t^*(x)$ is strictly monotone in $x$, by taking its inverse $\mathcal{X}(t,x)$, we can obtain $U_x(t,x)$ in terms of $\mathcal{Y}(t,x)$ and $\mathcal{X}(t,x)$.*

*Proof.* The first assertion of this result is essentially obtained by analogy with the primal problem. Indeed, using the $\tilde\beta$ expression's (23), which is

$$\hat\beta(t,y) = y\tilde U_y(t,y) r_t + \frac{1}{2} y^2 \tilde U_{yy}(t,y) \sup_{\nu_t \in \mathcal{R}^{\sigma,\perp}} \{-||\nu_t - \eta_t^\sigma||^2 - 2\big(\nu_t - \eta_t^\sigma\big).\big(\frac{\tilde\gamma_y(t,y)}{y\tilde U_{yy}(t,y)}\big)\}$$

One can easily remark, by analogy to expression of $\mathcal{Q}$ in Lemma 2.3 and that of $\beta$ (equation (13), Theorem 2.4), that $\tilde{U}$ is consistent with the family of processes $\mathscr{Y}$, that is $\tilde{U}(t, Y_t^{\nu})$ is a submartingale for any $\mathcal{Y}^{\nu} \in \mathscr{Y}$ and a local martingale for the optimal choice (Theorem 3.4) $\nu_t^*(y) = -\tilde{\gamma}_y^{\perp}(t,y)/y\tilde{U}_{yy}(t,y) = \gamma_x^{\perp}\big(t, -\tilde{U}(t,y)\big)/y$, if there exists a solution to the SDE,

$$\frac{dY_t^{\nu^*}}{Y_t^{\nu^*}} = -r_t dt + \big(\eta^{U,\perp}\big(t, -\tilde{U}_y(t, Y_t^{\nu^*})\big) - \eta_t^{\sigma}\big).dW_t. \tag{25}$$

On the other hand we recall that according to Proposition 2.6 assertion *ii)* $U_x(t, X_t^*)$ satisfies

$$\frac{dU_x(t,X_t^*)}{U_x(t,X_t^*)} = -r_t dt + \Big(\eta_t^{U,\perp}(X_t^*) - \eta_t^{\sigma}\Big).dW_t,$$

Note that $Y_t^*(y) = \Big(U_x(t, X_t^*(-\tilde{u}_y(y)))\Big)_{t\geq 0}$ and that $-\tilde{U}_y(t, Y_t^*(y)) = X_t^*(-\tilde{u}_y(y))$ shows that that $Y^*$ is a solution of (25) which in turn implies the optimality of $Y^*$.
To conclude, we have to show that $\tilde{U}(Y_t^*(y))$ is not only a local martingale but a "true" martingale, when $U(X_t^*(x))$ is a martingale. Put $x_y = -\tilde{u}_y(y)$, and use that the conjugacy relation implies that $\tilde{U}(Y_t^*(y)) = U(X_t^*(x_y)) - Y_t^*(y)X_t^*(x_y)$ with $Y_t^*(y) = \big(U_x(t, X_t^*(x_y)\big)$. Thanks to Proposition 2.6, $U(X_t^*(x_y))$ and $Y_t^*(y)X_t^*(x_y)$ and therefore $\tilde{U}(Y_t^*(y))$ are martingales. $\square$

**Decreasing Consistent Utilities** An interesting class of consistent utilities is the class of decreasing consistent utilities, which was studied and fully characterized in the literature by Berrier & al. [10] and Musiela & al. [35]. The utilities have a volatility vector $\gamma$ identically zero. It is an example where the dual SPDE is easier to study than the primal one. Indeed, taking $\gamma = 0$, $U$ is a solution of a non linear SPDE

$$dU(t,x) = \big[\frac{1}{2}\frac{U_x(t,x)^2}{U_{xx}(t,x)}||\eta_t^{\sigma}||^2\big]dt$$

where the convex conjugate $\tilde{U}$ satisfies

$$d\tilde{U}(t,y) = \big[-\frac{1}{2}y^2\tilde{U}_{yy}(t,y)||\eta_t^{\sigma}||^2 + r_t y\tilde{U}_y(t,y)\big]dt$$

Writing that $\frac{d\tilde{U}(t,y)}{dt} = U_t(t,y)$ yields

$$\tilde{U}_t(t,y)(\omega) = -\frac{1}{2}y^2\tilde{U}_{yy}(t,y)(\omega)||\eta_t^{\sigma}(\omega)||^2 + r_t(\omega)y\tilde{U}_y(t,y)(\omega)$$

which implies, by convexity, that $t \mapsto \tilde{U}(t,y)$ is a decreasing function. Moreover, it is easy to recognize in this PDE that the right hand side of the equation is nothing other than the operator

of diffusion of a geometrical Brownian motion with coefficients $\eta^\sigma_t(\omega)$ and $-r_t(\omega)\, L^{GB}_{t,y}(\omega)$ applied to $\tilde U$: $\tilde U_t(t,y)(\omega) = -L^{GB}_{t,y}\tilde U(t,y)(\omega)$. From this point, the idea is to look for positive solutions which are space-time harmonic functions of a geometric Brownian motion. As in Musiela & al and Berrier & al, the function $\tilde U(t,y)$ is assumed to be of class $\mathcal{C}^3$ in $y$ and of class $\mathcal{C}^1$ in time. First, put $\tilde V(t,y) = \tilde U(t, e^{-\int_0^t r_s ds} y)$ and $A(t) = \int_0^t ||\eta^\sigma_s||^2 ds$; then $\tilde V$ is a solution of the following PDE

$$\tilde V_t(t,y) = -\frac{1}{2}A_t(t)y^2\tilde V_{yy}(t,y).$$

Second, define $H(log(y) - \frac{1}{2}A_t, \frac{1}{2}A_t, \omega) = -\tilde V_{t,y}(t,y,\omega)$ and take the change of variable $\tau = \frac{1}{2}A_t$, it is straightforward to check that $H$ solves the backward heat equation,

$$H_\tau(\tau,z) + H_{zz}(\tau,z) = 0$$

The solutions of such equation are called space-time harmonic functions. Since the function $H$ is strictly positive, using the result of Widder, D.V [43, 44], F. Berrier & al. [10] and Musiela & al. [35] show the following result which characterizes all decreasing consistent utilities

**Theorem 3.6.** *Let $U(t,x)$ be a regular random field of class $\mathcal{C}^3$ on $x$ such that $\frac{\partial^2}{\partial y \partial t}\tilde U(t,y) = \frac{\partial^2}{\partial t \partial y}\tilde U(t,y)$ is defined and continuous. Assume $U$ satisfies the utility SPDE with $\gamma = 0$ a.s.. Then $U$ is a consistent stochastic utility if and only if there exists a constant $C \in \mathbb{R}$ and a finite Borel measure $m$, supported on the interval $(0,+\infty)$ with everywhere finite Laplace transform, such that*

$$\begin{aligned} \tilde U(t,y) &= \int_{\mathbb{R}^*_+} \frac{1}{1-\frac{1}{\alpha}}\big(1 - y^{1-\frac{1}{\alpha}} e^{-\frac{1-\alpha}{2\alpha}\int_0^t ||\eta_s||^2 ds}\big) dm(\alpha) + C. \\ \tilde U_y(0,y) &= -\int_{\mathbb{R}^*_+} y^{-\frac{1}{\alpha}} dm(\alpha) \end{aligned}$$

*Moreover the optimal wealth process is strictly increasing and regular with respect to its initial condition $x$.*

There is an interesting interpretation of these stochastic utilities: at date $t = 0$ the derivative $\tilde U_y(0,y) = u_y(y)$ cqscccan be easily interpreted as the integral $-y^{-\frac{1}{\alpha}}$ weighted by the measure $m$, which is nothing than the derivative of the convex conjugate of power utility with risk aversion $\alpha$. Hence, one can imagine that the investor starts from a power utility and pulls at random the risk aversion $\alpha$. At time 0, his dual utility is a mixture of power dual utilities weighted by some

measure $m$, and at time $t$, the same property is preserved but the measure becomes stochastic with density with respect to $m$, $m_t(d\alpha) := e^{-\frac{1-\alpha}{2\alpha}\int_0^t ||\eta_s||^2 ds} dm(\alpha)$.

The stochastic measure $m_t(d\alpha)$ is the unique one which ensure that the process $\tilde{U}$ constructed is the conjguate of a consistent utility. This interpretation is the starting point of the work [22] where more general method to construct consistent utilities processes from a family of classical utilities functions is developed.

## 3.3 Change of numeraire

One of our first reasons of interest in progressive utilities was the fact they are consistent with classical transformation in financial market in contrast to the classical utilities functions which are not stable by change of numeraire; so, the value function of the classical portfolio optimization problem is highly dependent on market parameters $(r, \eta)$. Moreover, it is easier to work with portfolios that are local martingales rather than semimartingales, that can be obtained using the market numéraire and there is a genuine interest to provide details of this transformation on consistent stochastic utilities.

Let's start with the following general result which is a direct consequence of the definition 1.1. In a more specific market: the bownien market which is the framework of this paper we can say more about the properties of these utilities in particular we can study the dynamics of these random fields as we see in next results.

**Proposition 3.7** (Stability by change of numeraire ).
*Let $U(t,x)$ be a $\mathscr{X}$-consistent stochastic utility, $N$ be a positive semimartingale and denote by $\hat{\mathscr{X}}$ the class of process defined by $\hat{\mathscr{X}} = \{\hat{X} := X/N,\ X \in \mathscr{X}\}$, then the process $V$ defined by*

$$V(t,x) = U(t, xN_t)$$

*is a $\hat{\mathscr{X}}$-consistent stochastic utility in the market of numeraire $N$ if and only if $U$ is an $\mathscr{X}$-consistent stochastic utility. Moreover, the optimal wealth processes are related by the following identity: $\hat{X}^* = X^*/N$.*

Roughly speaking the proposition says that the notion of $\mathscr{X}$-consistent utilities is preserved by change of numeraire. If the agent decides to invest in a secondary market (foreign market)

his preferences (risk aversion) are still unchanged, given the uniqueness representation of his preferences.

To show this result it is enough to verify the assertions of Definition 1.1 using identity $V(t,\hat{X}_t) = U(t,X_t)$ and the fact that by definition the optimal wealth processes are related by $\hat{X}^* = X^*/N$.

Now, we turn to more quantitative aspects of the change of numéraire. In the brownien market, the idea is to show how, by change of numeraire techniques, we simplify the utility SPDE's of consistent stochastic utilities. The local characteristics of the new stochastic utility $V$ are obtained by applying Ito-Ventzel Lemma and the parameters of the new $N$-market.

**Assumption 3.1.** *The new numeraire $N$, is assumed to satisfy:*

$$\frac{dN_t}{N_t} = \mu_t^N dt + \delta_t^N.dW_t,\ N_0 = z.$$

The wealth process $\hat{X}$ is defined by $\hat{X}_t := X_t/N_t$ where $X$ denotes the wealth process in the initial market. By Itô's formula, we can easily write the dynamics of $\hat{X}^\kappa = X_t^\kappa/N_t$,

$$\frac{d\hat{X}_t^\kappa(\hat{x})}{\hat{X}_t^\kappa(\hat{x})} = \big(r_t - \mu_t^N + \delta_t^{N,\sigma}.\eta_t^\sigma\big)dt + \big(\kappa_t - \delta_t^N\big).\big(dW_t + (\eta_t^\sigma - \delta_t^N)dt\big),\ \hat{x} = \frac{x}{z}.$$

Denoting $\hat{r} = r - \mu^N + \delta^{N,\sigma}.\eta^\sigma$ the short interest rate in the new market and by $\hat{\eta} = \eta^\sigma - \delta^N$ ($\hat{\eta}^\sigma = \eta^\sigma - \delta^{N,\sigma}$) the new market price of risk, we get

$$\frac{d\hat{X}_t^\kappa(\hat{x})}{\hat{X}_t^\kappa(\hat{x})} = \hat{r}_t dt + \big(\kappa_t - \delta_t^N\big).\big(dW_t + \hat{\eta}_t dt\big),\ \kappa_t \in \mathcal{R}_t^\sigma.$$

Let us now stress that if $\delta^N \in \mathcal{R}^\sigma$ the volatility vector $\hat{\kappa} = \kappa - \delta^N$ of $\hat{X}$ belongs to $\mathcal{R}^\sigma$. Hence we get that a consistent utility $V$ in this new market satisfies the same dynamics as $U$ in the initial market only by replacing $r,\ \eta$ by $\hat{r},\ \hat{\eta}$.

Else if $\delta^{N,\perp} \neq 0$ the optimization problem is quite different and utility-SPDE is modified, as we will see in the following result.

**Theorem 3.8.** *Let $U(t,x)$ be a $\mathscr{X}$-consistent stochastic utility satisfying the assumptions of Theorem 2.4. The $\hat{\mathscr{X}}$-consistent stochastic utility $V(t,x) := U(t,xN_t)$ is a solution of the following stochastic partial differential equation*

$$dV(t,x) = V_x(t,x)\Big\{\frac{1}{2\alpha^V(t,x)}\Big(||\hat{\eta}_t + \hat{\eta}_t^V||^2 - ||\hat{\eta}_t^\perp + \hat{\eta}_t^{V,\perp} - x\alpha^V\delta_t^{N,\perp}||^2\Big) - x\hat{r}_t\Big\}(t,x)dt + \gamma^V(t,x).dW_t$$

*with $\alpha^V(t,x) := V_x(t,x)/V_{xx}(t,x)$ and $\eta^V(t,x) = \gamma^V(t,x)/V_x(t,x)$ denote the risk tolerance and the utility risk premium of $V$. The volatility of $V$ is $\gamma^V(t,x) = \gamma^U(t,x\,N_t) + xV_x(t,x)\delta_t^N$ and the optimal policy $\hat{\kappa}^*$ is given by*

$$x\hat{\kappa}_t^*(x) = -\frac{1}{\alpha^V(t,x)}\Big(\hat{\eta}_t^\sigma + \eta_t^{V,\sigma}\Big)(t,x) \tag{26}$$

Let us comment on the content of this theorem and its relation to the previous results. The dynamic (26) of consistent utilities and the optimal policy (26) are more complicated then the ones in the initial framework. We recognize in the optimal policy formula a first term (very similar to that of the initial market) which corresponds to an optimization program without $\delta^N$ added to a second one which correspond to a translation by $\delta^N$. This is due to the fact that the dynamic of new wealth processes are a kind of combination of the initial market and the dynamics of the state price density processes in the dual problem studied in previous section. According to Remark 2.1, which generalizes Lemma 2.3, and due to the fact that volatilities of new wealth processes $\hat{X}$ are translated by the vector $-\delta^N$, the optimal policy in this new market, denoted by $\hat{\kappa}$ is given by

$$x\hat{\kappa}_t^*(x) = -\frac{1}{\alpha^V(t,x)}\Big(\hat{\eta}_t^\sigma + \eta_t^{V,\sigma}\Big)(t,x)$$

On the other hand, it suffices to take $\delta^N$ in the range of the matrix $\sigma$, to get the SPDE's of the old market and to take $\hat{\eta} = 0$ to get SPDE similar to the dual HJB-SPDE.

**Martingale market** Now, remark that there exists a market numeraire portfolio $X_t^{\eta^\sigma} = 1/Y_t^0 \in \mathscr{X}$ also called growth optimal portfolio see E. Platen and D. Heath [36] , which transforms the classical wealth processes into positive local martingales, by translating the space of constraints $\mathcal{R}^\sigma$. When using any state price density process $Y^\nu$ and the associated numeraire $N_t^\nu = 1/Y_t^\nu$, the local martingale property holds true, but the space of constraints is modified.

**Corollary 3.9.** *Under same assumptions as Theorem 3.8, taking $N$ equal to the numeraire portfolio $1/Y^0$, the market has no risk premium and the ratio $\eta^{V,\sigma}$ has the same impact as a risk premium, but depending on the level of the wealth $x$ at time $t$. In particular, the previous dynamic of $V$ is simpler*

$$dV(t,x) = \frac{(V_x)^2(t,x)}{2V_{xx}(t,x)}\|\frac{\gamma_x^{V,\sigma}(t,x)}{V_x(t,x)}\|^2 dt + \gamma^V(t,x).dW_t.$$

*and the convex conjugate $\tilde{V}$ of $V$ satisfies*

$$d\tilde{V}(t,y) = \frac{1}{2\tilde{V}_{yy}(t,y)}\|\tilde{\gamma}_y^{\perp}(t,y)\|^2 dt + \tilde{\gamma}^V(t,y).dW_t.$$

*Where the optimal policies are given by:*

$$x\hat{\kappa}_t^*(x) = -\frac{\eta^{V,\sigma}(t,x)}{\alpha^V(t,x)} = -\frac{\gamma^{V,\sigma}(t,x)}{V_{xx}(t,x)},\ \hat{\nu}_t^*(y) = -\frac{\tilde{\gamma}_y^{V,\perp}(t,y)}{\tilde{U}_{yy}(t,y)}$$

In this result, taking a numeraire with good properties the HJB-SPDE is simplified and become more intuitive. The utility process is a supermartingale and its dual conjugate is a submartingale. Moreover, the optimal policies are only characterized and driven by the derivative of the volatility vector.
We end this section, by the following corollary which is a consequence of the above result.

**Corollary 3.10.** *Under the assumptions of Theorem 3.8, taking $N = Y^0$, we have*

- $\gamma_x^V \in \mathcal{R}^\sigma$ *implies that $\tilde{V}$ is a local martingale and the optimal dual process is constant: $Y^* \equiv 1$.*
- $\gamma_x^V \in \mathcal{R}^{\sigma,\perp}$ *implies that $V$ is a local martingale and the optimal wealth $X^*(x) \equiv x$.*

The new market defined from the first one by change of numeraire $X^{\eta^\sigma}$ (the market numeraire) is called a martingale market because new wealths are local martingales.

# 4 Utility Characterization and Stochastic Flows Method for Solving Utility-Stochastic PDE's

As it is mentioned earlier, conventional methods of resolution of SPEs defined from their terminal conditions, such as the method of characteristics, can not be used to solve utility SPDEs. To overcome this difficulty, we present here a new approach for these utility stochastic partial differential equations, based on a random change of variable. This new approach, that we call "stochastic flows method", is based on the properties of the optimal wealth $X^*$ and the state price density process $Y^*$, both supposed to be a monotonic function of their initial condition. More precisely, we take the initial condition $U(0,.) = u(.)$ and an admissible wealth process as given, and ask: what are the conditions on these data to be an optimal wealth process of

some consistent stochastic utility $U$, and also how to recover $U$ from this information? In the classical expected utility framework, this is the question that He and Huang [12] asked in (1992) in a complete market. From one point of view, our problem is easier to solve because we allow ourselves a larger class of utility functions. In particular, we establish in the following that the only restriction is the monotony of the wealth process with respect to the initial wealth, plus some integrability condition. Another difference between this work and that of He and Huang [12] is that we work directly on the path of wealth process while they work with the volatility of the wealth process $\kappa(t,x)=\kappa(t,S_t,x)$ in their setup.
Before presenting this new method, remember that the direct analysis gives us a natural way of finding $U$ from the inputs $(u,Y^*,X^*)$ : Let $U$ be a consistent utility with optimal wealth $X^*$ then, according to Theorem 3.5 and Proposition 2.6, the process $Y^*$ defined by $Y_t^*(u_x(x)) = U_x(t,X_t^*(x))$ is optimal for the dual problem and such that $Y^*X^*$ is a martingale. So, if $X^*$ is strictly increasing with respect to the initial capital, with inverse flow $\mathcal{X}$, $U_x(t,x) = Y_t^*(u_x(\mathcal{X}(t,x)))$; integrating with respect to $x$ we get $U$.
From this we assume for the rest of the paper the following main assumption.

**Assumption 4.1.** *The wealth process $X_t^*(x)$ is assumed to be continuous and increasing in $x$ from $0$ to $+\infty$ with $X_t^*(0)=0$, $\;X_t^*(+\infty)=+\infty$ for any $t$ and satisfies*

$$\frac{dX_t^*(x)}{X_t^*(x)} = r_t dt + \kappa_t^*(X_t^*(x)).\big(dW_t + \eta_t^\sigma dt\big), \quad \kappa_t^*(x)\in\mathcal{R}_t^\sigma,\ \forall x>0,\ a.s.$$

*Denote by $\mathcal{X}(t,z)$ the inverse flow such that $X_t^*(\mathcal{X}(t,z)))=z$.*

Financially speaking this hypothesis, which may be a consequence of a no arbitrage opportunity, says that: we do not invest more to earn less. On the other hand, this monotony assumption is true in many examples and, according to the classical results of stochastic differential equations (SDE), is satisfied as soon as $x\kappa_t^*(x)$ is locally uniformly Lipschitz, (see Kunita [25]).
Note also, by conjugacy identity, that monotonicity of $X^*(x)$ implies that the dual process $Y^*(y)$ is, in turn, strictly increasing and therefore invertible with respect to its initial condition $y$ for any date $t$. The converse property is also true. Consequently, we also have the following hypothesis.

**Assumption 4.2.** $Y_t^*(y)$ *is continuous and increasing in* $y$ *from* $+\infty$ *to* $0$ *satisfying*

$$\frac{dY_t^*(y)}{Y_t^*(y)} = -r_t dt + \big(\nu_t^*(Y_t^*(y)) - \eta_t^\sigma\big).dW_t, \quad \nu_t^*(y) \in \mathcal{R}^{\sigma,\perp},\ \forall y > 0,\ a.s. \tag{27}$$

Starting from the idea above, Assumptions 4.1 and 4.2 allow us to compose $X^*$ with the inverse of $Y^*$ and $Y^*$ with the inverse flow of $X^*$. Under some additional regularity assumptions, we establish one of our main contribution that involves the characterization of any consistent utilities generating $X^*$ as an optimal portfolio. In particular, we give the decomposition of the derivative $\gamma$ of the volatility vector as an operator of $U_x$ and $U_{xx}$ given $\kappa^*$. The second main result of this paper introduce a new method to solve the utility stochastic PDE. The idea is to transform the SPDE in a system of two stochastic differential equations (SDE). Herein, the method proposed can be used for a large class of SPDE.

There are two different messages in our approach hence we decide to present the associated results separately. Note that the results of this sections can be obtained first on the martingale market and, simply, by using the results of Theorem 3.7 we get simillar ones on the initial market.

## 4.1 Utility Characterization from optimal wealth and state density processes

To fix the idea we consider a given wealth process $X^*$, a state density price process $Y^*$ and an utility function $u(x)$. The objective is to construct a consistent utility $U$ starting from the function $u(x)$ ($U(0,x) = u(x)$), generating $X^*$ as optimal wealth and $Y^*$ as optimal dual process. According to the necessary analysis above, the constructed utility process satisfies $U_x(.,X^*(x)) = Y^*(u_x(x))$.

**Remark .** *Before continuing our investigations it is important to note that the duality is not needed in what follows. By Proposition 2.6 assertion ii),* $U_x(.,X^*(x))$ *is a density price process, which is sufficient to present our new approach.*

**Linear optimal state density process.** To illustrate our approach we first start by proving this result in a special case where we assume that the process $Y^*(y) = yY^0$ since $\nu^* = 0$ a.s. The advantage of this case is that we can find all messages of our construction

and a complete overview of the main calculations. This will better the understanding of the calculations that are made in the general case, because they are more difficult to follow, although not really more complicated.

Assuming $Y^*(y) = yY^0$, the identity $Y_t^*(y) = U_x(t, X_t^*(-\tilde{u}_y(y))$ (Theorem 3.5) suggests a very simple way to associate a progressive utility $U(t,x)$ with the wealth process $X^*$. Indeed, if $\mathcal{X}(t,z)$ is the inverse flow of $X_t^*(x)$, then the increasing process $U(t,x)$ satisfying $U_x(t,x) = u_x(\mathcal{X}(t,x))Y_t^0$ is a good candidate to be a consistent stochastic utility. Another remarkable property of this random field is that $U_x(t, X_t^*(x)) = U_x(0,x)Y_t^0$, which is another way to express that the optimal dual policy $\nu^*$ is null. We are then ready to state one of the important results of this section.

**Theorem 4.1.** *In addition to the monotony assumption 4.1, assume that the given admissible portfolio $(X_t^*(x))$ has a volatility $\kappa_t^*(X_t^*)$, where the process $\kappa_t^*(x)$, is sufficiently regular to make the process $X_t^*(x)$ a $\mathcal{C}^{(1)}$ regular in the sense of Kunita. In addition, we assume that $\big(Y_t^0 X_t^*(x)\big)$ is a martingale $\forall x > 0$. Recall that $\mathcal{X}$ is the inverse flow of $X$.*
*Let $u$ be a utility function such that $x \mapsto u_{xx}(x)X_t^*(x)$ is integrable near to infinity. Then we define the processes $U$ and $\tilde{U}$ by,*

$$U(t,x) = Y_t^0 \int_0^x u_x(\mathcal{X}(t,z))dz, \quad \tilde{U}(t,y) = \int_y^{+\infty} X_t^*(-\tilde{u}_y(\frac{z}{Y_t^0}))dz. \tag{28}$$

*$U$ is a progressive utility, whose convex conjugate is $\tilde{U}$, and satisfies the dynamics*

$$\begin{cases} dU(t,x) &= \Big( -U(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||\gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma||^2\Big)dt + \gamma(t,x).dW_t \\ \gamma_x(t,x) &= -U_{xx}(t,x)x\kappa_t^*(x) - U_x(t,x)\eta_t^\sigma \end{cases}$$

*$\tilde{U}(t, yY_t^0)$ and $U(t, X_t^*)$ are martingale processes and $U$ is a $\mathscr{X}$-consistent stochastic utility, with optimal wealth $X^*$.*

Note the fact that the state price density process $Y^*(y) = yY^0$ is linear with respect to its initial condition greatly simplify this first result (true if $\nu^*$ is not depending on $y$) contrary to the next theorem where $\nu_t^*$ is a function of $y$.

*Proof.* First by definition $U$ is strictly increasing concave random field and of class $\mathcal{C}^{(3)}$ in the sense of Kunita. Let us now focus on the dynamics of this progressive utility. Any thing is

simpler in a martingale market, when $Y_t^0$ is replaced by 1, but the PDEs are a little more complicated.

To get started, we introduce the intermediate process $\bar{U}(t,x) := \int_0^x u_x(\mathcal{X}(t,z))dz$ with a simpler convex conjugate $\tilde{\bar{U}}(t,y) = \int_y^{+\infty} X_t^*((u_x)^{-1}(z))dz$. Denoting by $(\beta^{\bar{U}}, \gamma^{\bar{U}})$ and $(\tilde{\beta}^{\tilde{\bar{U}}}, \tilde{\gamma}^{\tilde{\bar{U}}})$ the local characteristic of $\bar{U}$ and $\tilde{\bar{U}}$, it follows from the dynamics of $X^*$ and the identity $\tilde{\bar{U}}_y(t,y) = -X_t^*(-\tilde{u}_y(y))$ that

$$\begin{cases} \tilde{\beta}_y^{\tilde{\bar{U}}}(t,y) &= -r_t\tilde{\bar{U}}_y(t,y) - \tilde{\bar{U}}_y(t,y)\kappa_t^*(-\tilde{\bar{U}}_y(t,y)).\eta_t^\sigma \\ \tilde{\gamma}^{\tilde{\bar{U}}}(t,y) &= -\tilde{\bar{U}}_y(t,y)\kappa_t^*(-\tilde{\bar{U}}_y(t,y)) \end{cases} \tag{29}$$

On the other hand, using the correspondence between the diffusion parameters of $\bar{U}$ and $\tilde{\bar{U}}_y$ given in Proposition 3.2 (or equivalently Corollary 3.3), we have

$$\begin{cases} \gamma_x^{\bar{U}}(t,x) &= -\bar{U}_{xx}(t,x)\tilde{\gamma}_y^{\tilde{\bar{U}}}(t,\bar{U}_x(t,x)) \\ \beta^{\bar{U}}(t,x) &= \tilde{\beta}^{\tilde{\bar{U}}}(t,U_x(t,x)) + \frac{1}{2\bar{U}_{xx}(t,x)}||\gamma_x^{\bar{U}}(t,x)||^2 \end{cases}$$

By this and (29), it is straightforward to check that

$$\begin{cases} \gamma_x^{\bar{U}}(t,x) &= -x\bar{U}_{xx}(t,x)\kappa_t^*(x) \\ \beta_x^{\bar{U}}(t,x) &= -x\bar{U}_{xx}(t,x)r_t + \gamma_x^{\bar{U}}(t,x).\eta_t^\sigma + \frac{\partial}{\partial x}\Big(\frac{1}{2\bar{U}_{xx}(t,x)}||\gamma_x^{\bar{U}}(t,x)||^2\Big) \end{cases}$$

In turn, we get that $\bar{U}(t,x)$ satisfies

$d\bar{U}(t,x) = \beta^{\bar{U}}(t,x)dt + \gamma^{\bar{U}}(t,x).dW_t$ with

$$\begin{cases} \gamma^{\bar{U}}(t,x) &= -\int_0^x \bar{U}_{xx}(t,z)z\kappa_t^*(z)dz \\ \beta^{\bar{U}}(t,x) &= -x\bar{U}_x(t,x)r_t + \bar{U}(t,x)r_t + \gamma^{\bar{U}}(t,x).\eta_t^\sigma + \frac{1}{2\bar{U}_{xx}(t,x)}||\gamma_x^{\bar{U}}(t,x)||^2 \end{cases}$$

As $U(t,x) = Y_t^0\bar{U}(t,x)$, Itô's formula leads to

$$dU(t,x) = Y_t^0\big(\beta^{\bar{U}}(t,x) - \bar{U}(t,x)r_t - \gamma^{\bar{U}}(t,x).\eta_t^\sigma\big)dt + \Big(Y_t^0\gamma^{\bar{U}}(t,x) - U(t,x)\eta_t^\sigma\Big).dW_t.$$

Denote by $\gamma(t,x) := Y_t^0\gamma^{\bar{U}}(t,x) - U(t,x)\eta_t^\sigma = -\int_0^x \bar{U}_{xx}(t,z)z\kappa_t^*(z)dz - U(t,x)\eta_t^\sigma$, we obtain using $\beta^{\bar{U}}$ formula and identities $U = Y^0\bar{U}$, $U_x = Y^0\bar{U}_x$, $U_{xx} = Y^0\bar{U}_{xx}$ that $U$ satisfies the desired dynamics given by

$$dU(t,x) = \Big(-U(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||\gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma||^2\Big)dt + \gamma(t,x).dW_t$$

and $\gamma_x(t,x) = -U_{xx}(t,x)x\kappa^*_t(x) - U_x(t,x)\eta^\sigma_t$.
Then $U$ is a progressive utility satisfying the utility non linear SPDE, with an optimal wealth satisfying equation (12) of Theorem 2.4. To conclude, it suffices to prove that $U(t,X^*_t)$ is a martingale. It is simpler to show this property on the conjugate dual process, since $\tilde{U}(yY^0_t) = Y^0_t\int_y^{+\infty} X^*_t(-u(z))dz$ is a martingale since $(Y^0_t X^*_t(z))$ is a martingale, by integrability assumption. By the conjugacy relation, the same property holds for $U(t,X^*_t)$. □

***Risk tolerance dynamics.*** With the utility characterization given in Theorem 4.1, the study of the risk tolerance coefficient, taken along the optimal wealth, is greatly simplified. In particular, the nice martingale property established in He and Huang in [12], in a complete market, may be generalized to consistent utilities.

**Proposition 4.2.** *Let $\alpha^U(t,x) = -\frac{U_x(t,x)}{U_{xx}(t,x)}$ be the risk tolerance coefficient of $U$.*
*Then $\alpha^U(t,X^*_t(x)) = \alpha^u(x)X^*_x(t,x)$, where $X^*_x(t,x)$ is the derivative (assumed to exist) of $X^*_t(x)$ with respect to $x$. Moreover, denoting $Y^*_y$ the partial dervative of $Y^*$ with respect to its initial condition, the process $Y^0_t\alpha^U(t,X^*_t(x)) \equiv Y^*_y(t,y)\alpha^U(t,X^*_t(x))$ is a local martingale, since $X^*_x(t,x)$ is also an admissible portfolio with initial wealth 1.*

*Proof.* Observe that by definition $U_{xx}(t,x) = Y^0_t u_{xx}(\mathcal{X}(t,x))\mathcal{X}_x(t,x)$. Since $\mathcal{X}_x(t,x) = 1/X^*_x(t,\mathcal{X}(t,x))$, and $\mathcal{X}(t,X^*(t,x)) = x$, the formula $\alpha^U(t,X^*_t(x)) = \alpha^u(x)X^*_x(t,x)$ is a simple verification. Moreover, observe that the derivative $X^*_x(t,x)$ (assumed to exist) belongs to the same vector space of processes than $X^*_t(z)$, and $Y^0_t X^*_x(t,x)$ is a local martingale. □

An extension of this result to the general framework where $Y^*$ is not necessarily linear on its initial condition is given by Proposition 4.4.

**General results.** We have shown in Theorem 4.1 that for a monotone wealth process $X^*$, Assumption ($X^*Y^0$ is a martingale) is sufficient in order to construct at least a consistent utility whose optimal wealth is $X^*$ and the optimal dual process $Y^*(y) = yY^0$. The extension to a general processes $Y^*$ ( not necessarily linear on its initial condition) is suggested by Theorem 3.5 which gives us the general form of consistent utilities using identity $U_x(t,x) = \mathcal{Y}\circ\mathcal{X}(t,x)$ where $\mathcal{Y}(t,x) = Y^*_t(u_x(x))$. Starting from this identity, we proceed by verification, the ideas are the same as in the proof of Theorem 4.1 but the calculations and equations are more complicated

due to the fact that the diffusion parameters of a general $Y^*$ depond on $Y^*$ and not necessarily in linear form.

The general utility characterization result is the following one.

**Theorem 4.3.** *Let $(X_t^*(x) \in \mathscr{X})$ be an admissible wealth process and $(Y_t^*(y)) \in \mathscr{Y}$ be an admissible state price density process, $\mathcal{C}^{(1)}$ regular in the sense of Kunita, such that in addition to Assumptions 4.1 and 4.2, $(X_x^*(t,x)Y_t^*(y))$ is a martingale, where by definition $X_x^*(t,x) = \partial_x X^*(t,x)$.*

*Let $u$ be a utility function, and $\mathcal{Y}(t,x) = Y_t^*(u_x(x))$, $\mathcal{X}(t,z) = (X_t^*(.))^{-1}$ two regular stochastic flows such that $x \mapsto \mathcal{Y}(t, \mathcal{X}(t,z))$ is integrable near to zero.*

*Define the processes $U$ and $\tilde{U}$ by*

$$U(t,x) = \int_0^x \mathcal{Y}(t, \mathcal{X}(t,z))dz, \quad \tilde{U}(t,y) = \int_y^{+\infty} X_t^*((\mathcal{Y})^{-1}(t,z))dz. \tag{30}$$

*$U$ is a progressive utility, whose convex conjugate is $\tilde{U}$, and the dynamics*

$$dU(t,x) = \Big( - xU_x(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||\gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma||^2\Big)dt + \gamma(t,x).dW_t,$$

*with volatility vector $\gamma$ given by*

$$\gamma(t,x) = -U(t,x)\eta_t^\sigma - \int_0^x \Big(zU_{xx}(t,z)\kappa_t^*(z) - \nu_t^*(U_x(t,z))\Big)dz.$$

*The associated optimal portfolio and the optimal dual process are $X^*$ and $Y^*$. Moreover $U(t, X_t^*)$ is a martingale, so that $U$ is a $\mathscr{X}$-consistent stochastic utility.*

In the first theorem of this section we built for a given initial utility function a consistent stochastic utility with a given optimal wealth process. The extension which we give here characterizes all consistent stochastic utilities with the same optimal wealth process. This result expresses only how we must diffuse the function $U_x(0,x) = u_x(x)$ to stay within the framework of consistent stochastic utilities in incomplete market. The answer is intuitive because it expresses the fact that it is enough to keep a monotone field $Y^*$ which does not influence the reference market. On the other hand it is important to remark that the derivative with respect to $x$ of the volatility vector $\gamma$ is the sum of two orthogonal vectors and is given by

$$\begin{aligned}\gamma_x(t,x) &= \nu_t^*(U_x(t,x)) - U_{xx}(t,x)x\kappa_t^*(x) - U_x(t,x)\eta_t^\sigma \\ &= \nu_t^*(U_x(t,x)) - U_x(t,x)\Big(\eta_t^\sigma + \frac{U_{xx}}{U_x}(t,x)x\kappa_t^*(x)\Big),\end{aligned}$$

and consequently, given $\kappa^*$ and $\nu^*$, it is interpreted as an operator $\Upsilon(t,x,U_x,U_{xx})$ which is linear on $U_{xx}$, that depends on $U_x$ through the volatility $\nu^*$ of $Y^*$ and an affine term on $\eta^\sigma$, and depends on $x$ only through the optimal policy $\kappa^*$. We also emphasize that the term $U_x/U_{xx}$ in this formula is the risk tolerance of an investor with utility process $U$. In particular, for the case of the market martingale ($\eta^\sigma = 0$), $\Upsilon(t,x,U_x,U_{xx})$ is linear on $U_{xx}$, depends on $U_x$ only through the volatility $\nu^*$ of $Y^*$ and on $x$ only through the optimal policy $\kappa^*$.
Note that in the classical backward set-up of utility maximization, a similar idea is investigated by I. Karatzas & al [9]. The authors show also that the solution of a backward SPDE can be represented as the composite of two invertible processes. But this differs from the approach proposed here because these processes are represented as an expectation of monotonic functions (characteristics method) where in this work are stochastic flows. Note that the authors also use the Itô-Ventzel formula to establish the backward SPDE.

**Remark .** *After giving the proof of this result, we want to draw the attention to the fact that this theorem can be showed first in the case of the martingale market* ($r=0,\ \eta^\sigma=0$)*. This allows us to simplify the calculations and then we can always comeback to the initial market by a technique of change of numeraire.*

*Proof.* Under Assumption 4.1 the inverse $\mathcal{X}$ of $X^*$ with respect to $x$ satisfies by Proposition 3.2

$$d\mathcal{X}(t,x) = -x\mathcal{X}_x(t,x)\kappa_t^*(x).dW_t + \Big[-x\mathcal{X}_x(t,x)r_t + \frac{1}{2}\partial_x\big(\mathcal{X}_x(t,x)\|x\kappa_t^*(x)\|^2\big)\Big]dt.$$

The hypothesis made on $X^*$ and $Y^*$ entails that we can apply the Itô-Ventzel formula to the compound flow $\mathcal{Y}\circ\mathcal{X}$. To study $U_x(t,x)$ we are first interested on the coefficient of $dW_t$ of the dynamics of $\mathcal{Y}\circ\mathcal{X}$ because it represents the derivative of the volatility $\gamma$ of the utility $U$. As $(\mathcal{Y}_x\circ\mathcal{X})\mathcal{X}_x = U_{xx}$ and $U_x = \mathcal{Y}\circ\mathcal{X}$, formula (3.1) gives us that

$$\gamma_x(t,x) = \nu_t^*(U_x(t,x)) - xU_{xx}(t,x)\kappa_t^*(x) - U_x(t,x)\eta_t^\sigma.$$

This identity shows that the vector $\gamma_x$ is the sum of two orthogonal vectors since the first term $\nu_t^*(U_x(t,x))$, by hypothesis, belongs to the orthogonal of the second which is $-U_x(t,x)\Big(\eta_t^\sigma + (xU_{xx}/U_x)(t,x)\kappa_t^*(x)\Big)$ that belongs by hypothesis to the space $\mathcal{R}_t^\sigma$. Throughout, the projection of $\gamma_x$ on $\mathcal{R}_t^\sigma$ is the vector $\gamma_x^\sigma(t,x) = -xU_{xx}(t,x)\kappa_t^*(x) - U(t,x)_x\eta_t^\sigma$.
As $U_x = \mathcal{Y}\circ\mathcal{X}$, $\gamma_x$ is the volatility process of $U_x$, it is enough to integrate it with respect to $x$ to obtain the result.

We now focus our interest on the drift $\mu^{U_x}$ of the derivative $\mathcal{Y}\circ\mathcal{X}$ of $U$. The idea and calculations are exactly identical to those of the proof of the previous result. Indeed by the assumptions and equation (20), we have

$$\begin{aligned}
&\mu^{U_x}(t,x) = -\big(x\mathcal{X}_x(t,x)\mathcal{Y}_x\circ\mathcal{X}(t,x) + \mathcal{Y}\circ\mathcal{X}(t,x)\big)r_t\\
&+\frac{1}{2}(\mathcal{Y}_x\circ\mathcal{X})(t,x)\partial_x\big(\mathcal{X}_x(t,x)\|x\kappa_t^*(x)\|^2\big) + \frac{1}{2}(\mathcal{Y}_x o\mathcal{X})(t,x)\|\mathcal{X}_x(t,x)x\kappa_t^*(x)\|^2\\
&-x\mathcal{X}_x(t,x)\partial_x\Big(\mathcal{Y}\circ\mathcal{X}(t,x)(\nu_t^*(\mathcal{Y}\circ\mathcal{X}(t,x)) - \eta_t^\sigma)\Big).\kappa_t^*(x) - x\mathcal{X}_x(t,x)\mathcal{Y}_x\circ\mathcal{X}(t,x)\kappa_t^*(x).\eta_t^\sigma.
\end{aligned}$$

Note that in the last line the term $-x\mathcal{X}_x(t,x)\partial_x\Big(\mathcal{Y}\circ\mathcal{X}(t,x)(\nu_t^*(\mathcal{Y}\circ\mathcal{X}(t,x)) - \eta_t^\sigma).\kappa_t^*(x)$ comes from Itô-Ventzel formula and corresponds to $< d\mathcal{Y}_x, d\mathcal{X} >$.
To lead the proof we proceed by analyzing line by line the above equality. Using the identities $(\mathcal{Y}_x\circ\mathcal{X}(t,x))_x = \mathcal{Y}_{xx}o\mathcal{X}(t,x)\mathcal{X}_x(t,x)$ and $U_{xx}(t,x) = \mathcal{Y}_x o\mathcal{X}(t,x)\mathcal{X}_x(t,x)$, the first line becomes

$$-\big(x\mathcal{X}_x(t,x)\mathcal{Y}_x\circ\mathcal{X}(t,x) + \mathcal{Y}\circ\mathcal{X}(t,x)\big)r_t = -\big(xU_{xx} + U_x\big)(t,x)r_t = -\partial_x(xU_x)(t,x)r_t. \quad (31)$$

Rewritting the second line, we obtain

$$\begin{aligned}
&\frac{1}{2}\Big[\big(\mathcal{Y}_x\circ\mathcal{X}(t,x)\mathcal{X}_x(t,x)(\mathcal{X}_x(t,x)\|x\kappa_t^*(x)\|^2\big) + (\mathcal{Y}_{xx}\circ\mathcal{X})(t,x)\mathcal{X}_x(t,x)\partial_x\big(\mathcal{X}_x(t,x)\|x\kappa_t^*(x)\|^2\big)\Big]\\
&= \frac{1}{2}\partial_x\big[\mathcal{Y}_x\circ\mathcal{X}(t,x)\mathcal{X}_x(t,x)\|x\kappa_t^*(x)\|^2\big]. \qquad (32)
\end{aligned}$$

Finally, from the assumption that $\nu_t^*(\mathcal{Y}(t,x)).\kappa_t^*(X_t^*(x)) = 0$, we deduce $\nu_t^*(\mathcal{Y}\circ\mathcal{X}(t,x)).\kappa_t^*(x) = 0$ and $\partial_x\big(\nu_t^*(\mathcal{Y}\circ\mathcal{X}(t,x))\big).\kappa_t^*(x) = 0$. This yields in the last line to

$$-x\mathcal{X}_x(t,x)\Big[\partial_x\Big(\mathcal{Y}\circ\mathcal{X}\big(\nu_t^*(\mathcal{Y}\circ\mathcal{X}) - \eta_t^\sigma\big)(t,x)\Big).\kappa_t^*(x) - \mathcal{Y}_x\circ\mathcal{X}(t,x)\kappa_t^*(x).\eta_t^\sigma\Big] = 0. \qquad (33)$$

Identities (31), (32) and (33) combined with the expression of $\mu^{U_x}$ and $\gamma_x$ yield to

$$\mu^{U_x}(t,x) = \partial_x\Big(-xU_x(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||x\kappa_t^*(x)||^2\Big).$$

As $U(t,0)\equiv 0$ we get by integration that $U$ satisfies

$$dU(t,x) = \{-xU_x(t,x)r_t + \frac{1}{2}U_{xx}(t,x)||x\kappa_t^*(x)||^2\}dt + \gamma(t,x).dW_t.$$

Using $\gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma = -xU_{xx}(t,x)\kappa_t^*(x)$ one easily sees that

$$dU(t,x) = \{-xU_x(t,x)r_t + \frac{1}{2}\Big[\frac{\|\gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma\|^2}{U_{xx}(t,x)}\Big]\}dt + \gamma(t,x).dW_t.$$

It remains to show that $U(t,X_t^*)$ is a martingale, given that the positive process $Y_t^*(y)X_x^*(t,x)$ is a martingale by assumption. Then since $U(t,X_t^*) = \int_0^x \mathcal{Y}(t,z)X_x^*(t,z)dz$, $U(t,X_t^*)$ is also a martingale. The proof is complete. $\square$

For more general processes $Y^*$, as in Proposition 4.2, the risk tolerance coefficient of $U$ taken along the optimal wealth has a nice properties.

**Proposition 4.4.** *Under the same assumptions as in Theorem 4.3, the risk tolerance coefficient $\alpha^U$ of $U$ is given by*

$$\alpha^U(t,x) = \frac{\mathcal{Y}\circ\mathcal{X}(t,x)}{\mathcal{Y}_x\circ\mathcal{X}(t,x)} X_x^*\circ\mathcal{X}(t,x).$$

*Where , $\mathcal{Y}(t,x) := Y_t^*(u_x(x))$. Moreover, $\alpha^U(t,X_t^*(x)) = \frac{Y_t^*(u_x(x))}{Y_y^*(t,u_x(x))u_{xx}(t,x)} X_x^*(t,x)$ and satisfies: $Y_y^*(t,y)\alpha^U(t,X_t^*(x))$ is a local martingale.*

## 4.2 Stochastic Flows Method for Solving Stochastic PDE's

In the previous section using two invertible stochastic flows $X^*$ and $Y^*$ we construct a consistent utility with the desired dynamics. Naturally, the question of the converse point of view is required. Starting from a stochastic PDE that satisfy consistent utilities, the question is then under which assumptions we have existence and uniqueness of the solution? What can we deduce about the monotony and the concavity of a possible solution? Answering these questions is the purpose of this section. In the following theorem we propose a new method that allows us to address the issue of such resolution of fully nonlinear second order stochastic PDEs .

**Theorem 4.5.** *Consider a utility stochastic PDE with initial condition $u(.)$,*

$$dU(t,x) = \Big( -xU_x(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||\gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma||^2\Big)dt + \gamma(t,x).dW_t. \qquad (34)$$

*Where the derivative $\gamma_x$ of $\gamma$ is the operator given by*

$$\gamma_x(t,x) = -U_x(t,x)\eta_t^\sigma - xU_{xx}(t,x)\kappa_t^*(x) + \nu_t^*(U_x(t,x)),\ \kappa_t^*\in\mathcal{R}_t^\sigma,\ \nu_t^*\in\mathcal{R}_t^{\sigma,\perp},\ t\geq 0.$$

*Assume that the both equations*

$$\frac{dX_t^*(x)}{X_t^*(x)} = r_t dt + \kappa_t^*(X_t^*(x)).\big(dW_t + \eta_t^\sigma dt\big), \quad \frac{dY_t^*(y)}{Y_t^*(y)} = -r_t dt + \big(\nu_t^*(Y_t^*(y)) - \eta_t^\sigma\big).dW_t \quad (35)$$

*admit solutions and that $X^*$ is monotonous: $[0,+\infty) \to [0,+\infty)$ in its initial condition and it is a regular flow in the sense of Kunita. Let $\mathcal{Y}(t,x) = Y_t^*(u_x(x))$, $\mathcal{X}(t,z) = (X_t^*(.))^{-1}$ and assume $x \mapsto \mathcal{Y}(t,\mathcal{X}(t,z))$ is integrable near to zero. Then there exists a solution $U$ of the SPDE (34) given by*

$$U(t,x) = \int_0^x \mathcal{Y}(t,\mathcal{X}(t,z))dz$$

*Moreover, if $X^*$ and $Y^*$ are increasing: $[0,+\infty) \to [0,+\infty)$ in their initial conditions and are a regular flows in the sense of Kunita, the random field $U$ is an increasing and concave solution of the SPDE (34). Finally, if $X^*$ and $Y^*$ are unique then $U$ is the unique solution of (34).*

Theorem 4.3 shows that for a given $X^*$ and $Y^*$ increasing solutions of SDEs (35) the random field $U(t,x) = \int_0^x \mathcal{Y}(t,\mathcal{X}(t,z))dz$ is a consistent utility solution of the utility SPDE (34) with a volatility vector $\gamma$ s.t. $\gamma_x(t,x) + U_x(t,x)\eta_t^\sigma = -xU_{xx}(t,x)\kappa_t^*(x) + \nu_t^*(U_x(t,x))$. In this result the converse point of view is investigated. Starting from the utility SPDE (34) with a given initial condition $u$, and a given $\kappa^*$ and $\nu^*$ such that $-xU_{xx}(t,x)\kappa_t^*(x) = \gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma$ and $\nu_t^*(U_x(t,x)) = \gamma_x^\sigma t,x)$, the theorem shows, under the assumption that both the SDE's (35) admit a solutions and if only $X^*$ is invertible then there exist a solution to the SPDE (34) given by $U(t,x) = \int_0^x \mathcal{Y}(t,\mathcal{X}(t,z))dz$. If both $X^*$ and $Y^*$ are strictly increasing regular flows the solution $U$ is increasing concave. Moreover, the uniqueness of $U$ is strongly related on the uniqueness of the solutions $X^*$ and $Y^*$ of SDE's (35). Remark that the martingale property of the product $X^*Y^*$ is not required to have a solution of the SPDE where it was necessary to conclude that $U$ is a consistent utility in Theorem 4.3. Finally, note that in previous sections if the random field $U(t,x)$ is a consistent regular utility then the processes $\kappa^*$ and $\nu^*$ ( $X^*$ and $Y^*$) exists and are regular.

This is an interesting new approach in which the solution of the utility SPDE have a trajectory (path wise) representation contrary to the characteristics method where the solutions are represented as an expectation. In particular, note that there are several advantages of this connection between SPDE's and SDE's. For example, the existence of diverse works in the domain of SDE's and seen in the multitude of results on the existence, uniqueness and on the integrability of solutions. The monotonicity of solutions $X^*$ and $Y^*$ gives several properties of the solution $U$ of the SPDE. To the best of our knowledge, there are no or few results that assert the monotonicity or the convexity of such solutions. Also, there may be other advantages in numerical methods and simulations of the SDE than of SPDE.

We finish this section mentioning that the main assumption of Theorem 4.5 is to assume that the SDEs (35) admit a solutions which is a fairly strong assumption because $\kappa^*$ and $\nu^*$ may depend on higher order derivatives of $U$. For example in the Markovian case where $U(t,x) = u(t,x,\Theta_t)$, according to Example 2.1, the volatility vector $\gamma$ of $U$ is given by

$$\gamma(t,x,\Theta_t) = \Sigma_t^{\Theta}\,\nabla_{\Theta} u(t,x,\Theta_t)$$

and the optimal startegy is given by

$$-x\kappa^*(t,x,\Theta_t) = \frac{\Sigma_t^{\Theta}\,\nabla_{\Theta} u_x(t,x,\Theta_t)}{u_{xx}(t,x,\Theta_t)}$$

In other words, for a given policy $\kappa^*$ the existence of a solution to the associated SDE in (35) is like an inverse problem.

Therefore, in the case where the optimal portfolio $X_t^*(x)$ is monotone with respect to its initial condition, we have by previous results the following characterization

$$u(t,x,\Theta_t) = \int_0^x Y_t^*\big(u_x(\mathcal{X}(t,z))\big)dz$$

and in particular $u(t,x,\Theta_t)$ is a solution of the stochastic PDE (34). But this is insufficient to characterize the function $u(t,x,\theta)$ which must satisfy an ordinary PDE of HJB type. So, to characterize $u(t,x,\theta)$, there is a further step that must be overcome. But, if in addition of monotony Assumption 4.1, $\theta \mapsto \Theta_t(\theta)$ is strictly increasing, then using the same method of stochastic change of variable we characterize $u(t,x,\theta)$ by inverting the global flow $(x,\theta) \mapsto \Big(X_t^*(x,\theta), \Theta_t(\theta)\Big)$. Then, we obtain

$$u(t,x,\theta) = \int_0^x Y_t^*\Big(u_x\big(0, \mathcal{X}(t,z,\Theta_t^{-1}(\theta)), \Theta_t^{-1}(\theta)\big), \Theta_t^{-1}(\theta)\Big)dz$$

Where $Y_.^* \equiv Y_.^*(y,\theta)$.

**Conclusion** This paper investigates consistent stochastic utilities from the stochastic PDEs point of view. This leads therefore to make strong regularity assumptions: The market is a Brownian market and securities are modeled as continuous semimartingales. Utilities are at least of class $\mathcal{C}^{(2)}$ in the sense of Kunita in order to apply Itô-Ventzel Lemma and to deduce the SPDEs. Moreover, the method of stochastic utilities construction is based on the dynamics of stochastic flows and their inverses, and therefore additional regularity assumptions on $X^*$ and

$Y^*$ are required. However, one can take a direct approach still based on monotony assumptions on optimal processes for the primal and dual problem, and on compound flows formula ; it is showed in [23], that these assumptions can be considerably weakened. Indeed, considering any financial market in which the securities are modeled as bounded semimartingales, the stochastic utilities are of class $\mathcal{C}^1$ and wealth process are required to lie in a convex class $\mathscr{X} \subset \mathbb{X}^+$, the monotony assumption of $X^*$ and $Y^*$ is sufficient to show the validity of the construction proposed in this work, using analysis methods and optimality conditions.

# Appendix

## A Itô-Ventzel's formula

The Itô-Ventzel's formula is a generalization of classical Itô's formula where the deterministic function is replaced by a stochastic process depending on a real or multivariate parameter. There are several difficulties in the definition of semimartingale depending on a parameter, as explained in H. Kunita [25]. For instance, let us consider the Itô integral of a predictable process $f_t(x)$ with parameter $x$ in some domain D of $\mathbb{R}^+$ with respect to some Brownian motion $B$. Suppose that $\int_0^T f_s(x)^2 ds < +\infty$ holds for each $x \in \mathrm{D}$. Then the Itô integral $M_t(x) = \int_0^t f_s(x) dB_s$ is well defined for any $t$ except for a null set $N_x$. It is a continuous local martingale with parameter $x \in \mathrm{D}$. Then $M_t(x)$ is well defined for $(t, x)$ if $\omega \in (\cup_{x\in\mathcal{I}} N_x)^c$. However the exceptional set $(\cup_{x\in\mathcal{I}} N_x)$ may not be a null set since it is an uncountable union of null sets. To overcome this technical problem we must take a good modification of the random field $M_t(x)$ so that it is well defined for all $(t, x)$ a.s. and is continuous or continuously differentiable with respect to $x$ for all $t$ almost surely.

### A.1 Notation and Definition

**Functional spaces** We shall first introduce some notations. Let D be a domain in $\mathbb{R}^+$, $m$ an non-negative integer and denote by $\mathcal{C}^m(\mathrm{D}, \mathbb{R})$ the set of all functions $g : \mathrm{D} \longrightarrow \mathbb{R}$ which are $m$-times continuously differentiable. Using the notation $g^{(m)}$ for the derivative of order $m$ of

some function $g$, we introduce the seminorms defined on some compact subset of D by

$$||g||_{m:K} = \sup_{x\in K} \frac{|g(x)|}{1+|x|} + \sum_{1\le\alpha\le m} \sup_{x\in K} |g^{(\alpha)}(x)|.$$

Equipped with these seminorms, $\mathcal{C}^m(\mathrm{D}, \mathbb{R})$ is a Frechet space. When D itself is a compact space we drop out the reference to $K$.

We sometimes need to refer to more regular functions whose derivatives of order $m$ are $\delta$-Hölder continuous ($0 < \delta \le 1$). Then we introduce a new family of seminorms,

$$||g||_{m+\delta:K} = ||g||_{m:K} + \sup_{\substack{x,y\in K \\ x\neq y}} \frac{|g^{(m)}(x) - g^{(m)}(y)|}{|x-y|^\delta}.$$

on the set of $\mathcal{C}^m(\mathrm{D}, \mathbb{R})$ whose last derivative is $\delta$-Hölder continuous.

**Definition A.1.** *A continuous function $f(t,x),\ x\in\mathcal{I},\ t\ge 0$ is said to belong to $\mathcal{C}^{m,\delta}, \delta\in[0,1]$ if for every $t$, $f(t) = f(t,.)$ belongs to $\mathcal{C}^{m,\delta}$ and $||f(t)||_{m+\delta:K}$ is integrable with respect to $t$ for any compact subset $K$ of $\mathcal{I}$. If the set $K$ is $\mathcal{I}$, the function $f$ is said to belong to the class $\mathcal{C}_b^{m,\delta}$. Furthermore, if $||f(t)||_{m+\delta}$ is bounded in $t$ it is said to belongs to $\mathcal{C}_{ub}^{m,\delta}$*

We also need to introduce the same kind of definition for functions depending on two parameters

$$||g||^{\sim}_{m+\delta:K} = ||g||^{\sim}_{m:K} + \sum_{\alpha=m} ||\partial_x^\alpha \partial_y^\alpha g(x,y)||^{\sim}_{\delta:K}$$

$$||g||^{\sim}_{\delta:K} = \sup_{\substack{x,x',y,y'\in K \\ x\neq x', y\neq y'}} \frac{|g(x,y) - g(x',y) - g(x,y') + g(x',y')|}{|x-x'|^\delta |y-y'|^\delta}.$$

**Definition A.2.** *A continuous function $g(t,x,y),\ x,y\in\mathcal{I},\ t\in[0,T]$ is said to belong to $\tilde{\mathcal{C}}^{m,\delta}, \delta\in[0,1]$ if for every $t$, $g(t) = g(t,.,.)$ belongs to $\tilde{\mathcal{C}}^{m,\delta}$ and $||g(t)||^{\sim}_{m+\delta:K}$ is integrable on $[0,T]$ with respect to $t$ for any compact subset $K$ of $\mathcal{I}$. If the set $K$ is $\mathcal{I}$, the function $g$ is said to belong to the class $\tilde{\mathcal{C}}_b^{m,\delta}$. Furthermore, if $||g(t)||^{\sim}_{m+\delta}$ is bounded in $t$ it is said to belong to $\tilde{\mathcal{C}}_{ub}^{m,\delta}$*

**$\mathcal{C}^{m,\delta}$-process :** Let $U(t,x)$ a family of real valued process with parameter $x\in\mathcal{I}$. We can regard it as a random field with double parameter $t$ and $x$. If $U(t,x,\omega)$ is a continuous function of $x$ for almost all $\omega$ for any $t$, we can regard $U(t,.)$ as a stochastic process with values in $\mathcal{C} = \mathcal{C}(\mathcal{I},\mathbb{R})$ or a $\mathcal{C}$-valued process. If $U(t,x,\omega)$ is $m$-times continuously differentiable with

respect to $x$ for almost all $\omega$ for any $t$, it can be regarded as a stochastic process with values in $\mathcal{C}^m = \mathcal{C}^m(\mathcal{I}, \mathbb{R})$ or a $\mathcal{C}^m$-valued process. If $U(t,x)$ is a continuous process with value in $\mathcal{C}^m$, it is called a continuous $\mathcal{C}^m$-process. A $\mathcal{C}^{m,\delta}$-valued process and continuous $\mathcal{C}^{m,\delta}$-processes are defined similarly.

$\tilde{\mathcal{C}}^{m,\delta}$**-process :** Let $G(t,x,y)$ be a stochastic process with parameter $x,\ y \in \mathcal{I}$. If it is $m$-times continuously differentiable with respect to each $x$ and $y$ a.s. for any $t$, it is called a stochastic process with values in $\tilde{\mathcal{C}}^m$ or a $\tilde{\mathcal{C}}^m$-valued process. The $\tilde{\mathcal{C}}^{m,\delta}$-valued process and continuous $\tilde{\mathcal{C}}^{m,\delta}$-valued process are defined similarly.

**Theorem A.1.** *Let $M_t(x),\ x \in \mathcal{I}$ be a family of continuous local martingales such that $M_0(x) \equiv 0$. Assume the joint quadratic variation $< M_t(x), M_t(y) >$ has a modification $A(t,x,y)$ of a continuous $\tilde{\mathcal{C}}^{m,\delta}$-process for some $m \geq 1$ and $\delta \in (0,1]$. Then $M_t(x)$ has a modification of a continuous $\mathcal{C}^{m,\varepsilon}$-process for any $\varepsilon < \delta$. Furthermore, for each $n \geq m$, $\partial_x^n M_t(x),\ x \in \mathcal{I}$ is a family of continuous local martingales with joint quadratic variation $\partial_x^n \partial_y^n A(t,x,y)$.*

**Definition A.3.** *We shall call the random field $M_t(x)$ with the property of the previous Theorem a continuous local martingale with values in $\mathcal{C}^{m,\varepsilon}$ or a continuous $\mathcal{C}^{m,\varepsilon}$-localmartingale.*

**Regular Itô's random fields** $\mathcal{C}^{m,\delta}$**-semimartingale :** Suppose $U(t,x),\ x \in \mathcal{I}$ is a family of continuous semimartingale decomposed as $U(t,x) = B(t,x) + M(t,x)$, where $M(t,x)$ is a local martingale and $B(t,x)$ is a continuous process of bounded variation. $U(t,x),\ x \in \mathcal{I}$ is said to belong to the class $\mathcal{C}^{m,\delta}$ or simply to be $\mathcal{C}^{m,\delta}$-semimartingale if $M(t,x)$ is a continuous $\mathcal{C}^{m,\delta}$-local martingale and $B(t,x)$ is a continuous $\mathcal{C}^{m,\delta}$-process such that $D_x^\alpha B(t,x),\ \alpha \leq m$ are all process of bounded variation. Further if $\delta = 0$ it is called a $\mathcal{C}^m$-semimartingale.

Let $U$ be a semimartingale satisfying

$$dU(t,x) = \beta(t,x)dt + \gamma(t,x).dW_t, \quad U(r,x) = U(x),$$

where $\beta$ and $\gamma$ are predictable process.

**Definition A.4** (Kunita)**.**

- *The pair $(\beta, \gamma)$ is called the local characteristic of $U$.*
- *Let $m$ be a non-negative integer and $a(t,x,y) := \gamma(t,x)^*\gamma(t,y)$. The local characteristic $(\beta, \gamma)$ is said to be in the class $\mathcal{B}^{m,0}$ if both $\beta$ and $a$ are predictable process with value $\mathcal{C}^m$ and if for any compact subset $K_1 \subset \mathbb{R}_+$ and $K_2 \subset \mathbb{R}_+ \times \mathbb{R}_+$ $||\beta(t,.)||_{m:K_1},\ ||a(t,.,.)||^{\sim}_{m:K_2} \in L^1$.*

Where

$$||f||_{m:K_1} = \sup_{x\in K_1} \frac{|f(x)|}{1+|x|} + \sum_{1\le|\alpha|\le m} \sup_{x\in K_1} |D_x^\alpha f(x)|.$$

$$||g||^{\sim}_{m:K_2} = \sup_{x,y\in K_2} \frac{|g(x,y)|}{(1+|x|)(1+|y|)} + \sum_{1\le|\alpha|\le m} \sup_{x,y\in K_2} |D_x^\alpha D_y^\alpha g(x,y)|$$

**Definition A.5** (Itô-Ventzel Regularity). *A semimartingale random field $U$ is said to be Itô-Ventzel regular if $U$ is a continuous $\mathcal{C}^2$-process and continuous $\mathcal{C}^1$-semimartingale with local characteristic satisfying previous assumption .*

**Theorem A.2** (Itô-Ventzel's Formula (Kunita)). *Let $(U(t,x))$ be an Itô-Ventzel regular semimartingale random field and let $X_t$ be a continuous semimartingale with values in $\mathcal{I}$ and volatility $\sigma^X$, then $U(t,X_t)$ is a continuous semimartingale and*

$$\begin{aligned} U(t,X_t) &= U(0,X_0) + \int_0^t \beta(s,X_s)ds + \int_0^t \gamma(s,X_s).dW_s \\ &+ \int_0^t \frac{\partial U}{\partial x}(s,X_s)dX_s + \int_0^t \frac{\partial^2 U}{\partial x^2}(s,X_s) < X >_s ds \\ &+ \int_0^t \langle \frac{\partial \gamma}{\partial x}(s,X_s), \sigma_s^X \rangle ds. \end{aligned}$$

Furthermore, according to H. Kunita [25] Theorem 3.3.3 p.94 we have the following differential rules for stochastic integrals.

**Theorem A.3** (Differential rules for stochastic integrals). *Let $F(t,x)$ be a continuous $\mathcal{C}^{m,\delta}$-semimartingale with local characteristic belonging to the class $\mathcal{B}^{m,\delta}$ where $\delta > 0$. Let $X(t,x)$, $x \in \Lambda$, $t \in [0,T]$ be a continuous predictable process with values in $\mathcal{C}^{k,\gamma}(\Lambda,\mathcal{I})$ where $\gamma > 0$ and $\Lambda \subset \mathbb{R}^e$. Set*

$$M(t,x) = \int_0^t F(X(s,x),ds).$$

*(i) Then $M(t,x)$ has a modification of continuous $\mathcal{C}^{m,\delta}$-semimartingale with values in $\mathcal{C}^{m\wedge k,\varepsilon}(\Lambda,\mathbb{R})$ with local characteristic belonging to the class $\mathcal{B}^{m\wedge k,\gamma\delta}$ with $\varepsilon < \gamma\delta$.*

*Further if $g_t$ is a continuous predictable process with values in $\Lambda$, then we have the equality:*

$$\int_0^t M(ds,g_s) = \int_0^t F(X(s,g_s),ds). \tag{36}$$

*(ii) If $m \geq 1$ and $k \geq 1$, then we have the equality:*

$$\frac{\partial}{\partial x^i} M(t,x) = \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x^j} X^i(s,x) \frac{\partial}{\partial x^i} F(X(s,x), ds). \tag{37}$$

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